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MOTION OF RIGID AND FLUID SPHERES IN STATIONARY
AND MOVING LIQUIDS INSIDE CYLINDRICAL TUBES

by

W.L. Haberman, Ph.D. and R.M. Sayre

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October 1958

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AND MOVING LIQUIDS INSIDE CYLINDRICAL TUBES**

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W.L. Haberman, Ph.D. and R.M. Sayre

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NOTATION

a	Constant of separation of partial differential equations
a_n, b_n	Constants in cylindrical coordinate solution
$A_n, B_n, C_n,$ D_n, E_n, F_n	Constants in spherical coordinate solution
b	Radius of cylinder
c_n	Constants
$C_n^{-1/2}(t)$	Gegenbauer polynomial
dS	Surface element
D	Drag of sphere
D_r	Normal drag of sphere
D_θ	Tangential drag of sphere
$f(r)$	Function of r
$f(\theta)$	Function of θ
$f(a), F(a),$ $g(a), G(a)$	Arbitrary functions in cylindrical coordinate solution
$F_n(t)$	Function of t
g	Gravitational constant
I_0	Modified Bessel function, first kind, zero order
I_1	Modified Bessel function, first kind, first order
k_n	Constants
K	Wall correction factor
K_0	Modified Bessel function, second kind, zero order
K_1	Modified Bessel function, second kind, first order
L	Operator $\left[\frac{\partial^2}{\partial r^2} + \frac{1 - \cos^2 \theta}{r^2} \frac{\partial^2}{\partial \cos \theta^2} \right]$ or $\left[\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial}{\partial \rho} \right]$
m	Constant
n	Integer
p	Pressure
p_{rr}	Stress component in r -direction
$p_{r\theta}$	Stress component in θ -direction

$P_n(t)$	Legendre polynomial
$Q(r)$	Function of r
$Q_n(t)$	Legendre function of second kind
r, θ	Spherical coordinates
R	Radius of sphere
$R(r)$	Function of r
$S(r)$	Function of r
S_1, S_2, S_3, S_4	Expressions defined by Equation [43]
S_2^a, S_3^a, S_4^a	Integrals defined on page 17
t	$\cos \theta$
$T(t)$	Function of t
u_r, u_θ	Velocity components inside sphere in spherical coordinates
U	Uniform velocity
v_r, v_θ	Velocity components in spherical coordinates
v_x, v_ρ	Velocity components in cylindrical coordinates
V	Maximum velocity of parabolic velocity distribution
x, ρ	Cylindrical coordinates
α	Order of various functions
α_n, β_n	Constants defined on page 15
γ	Density of external medium. Euler's constant
γ_i	Density of sphere
λ	Ratio of radii = $\frac{R}{P}$ or $\frac{R}{b}$
μ	Dynamic viscosity of external medium
μ_i	Dynamic viscosity of fluid inside sphere
P	Radius of spherical container
$P(\rho)$	Function of ρ
σ	Viscosity ratio = μ/μ_i
ϕ	Function defined in Equation [48]
ψ	Stream function

Subscript i Refers to quantities inside sphere

ABSTRACT

This report considers the problem of steady, axial translation of rigid and fluid spheres in stationary and moving viscous, incompressible fluids bounded by an infinitely long cylinder. The investigation is based on Stokes' approximation for the hydrodynamic equations for slow flow; thus inertia terms can be neglected, and the stream function satisfies a fourth-order differential equation similar in form to the biharmonic one.

An exact solution for the motion of rigid spheres in still and moving liquids within a cylindrical container has been obtained in terms of an infinite set of linear algebraic equations for the coefficients in the Stokes stream function. It is shown that the drag of a sphere in motion within a moving liquid is composed of two parts: namely, the drag due to the motion of the sphere in a still liquid inside the cylindrical tube, and the drag due to the motion of the liquid within the cylindrical tube past a stationary sphere. The drag experienced by the rigid spheres has been determined for the two special cases over a range of ratios of sphere-to-cylinder diameter. It is also shown that the first two equations of the infinite set closely approximate the motion and drag of rigid spheres over a large range of diameter ratios.

For fluid spheres (i.e., spheres which have different physical properties than the external medium and are characterized by internal motion) an approximate solution (similar to the one for the rigid case) is obtained. The drag experienced by the fluid spheres has been computed for the two special cases mentioned above for a range of diameter ratios.

Experimental results for the rigid and fluid case confirm the theory. In general, the results show that the wall effect for fluid spheres is less than for corresponding rigid spheres. Streamlines and velocity distributions for several cases where the diameter ratio is 0.5 are compared with those in an infinite medium.

INTRODUCTION

The effect of the proximity of the container walls on the drag of moving bodies is of interest in many fields of physics and engineering. Examples are the case of the falling ball viscosimeter, and the rise of air bubbles in tubes. The purpose of the present investigation is to determine the drag of a spherical body rising or falling in steady, slow motion (i.e., the inertia terms in the equation of motion can be neglected) in a liquid (stationary or moving) inside an infinitely long cylinder. Both rigid and fluid spheres (drops, air bubbles) are considered.

Previous investigations on the effect of the proximity of the container walls on the drag of moving bodies have dealt with rigid spheres.* Ladenburg¹ and Faxén² studied the drag of spheres moving in a still liquid contained in an infinitely long cylinder.** Wakiya⁵ and Happel and Byrne⁶ determined the drag of spheres in Poiseuille flow (parabolic velocity distribution) in a cylindrical tube. Wakiya considered the wall effect of a fixed rigid sphere in Poiseuille flow. Happel and Byrne also included the case where the rigid sphere is moving inside the cylinder. All used the method of reflection to obtain their solution. Starting with the known solution for the rigid sphere in an infinite medium, a "reflection" flow is superposed such that the boundary conditions on the cylinder are satisfied exactly. The boundary conditions on the sphere are only approximately satisfied. The drag of the spheres is obtained from Stokes' law using the velocity of the sphere increased by the average "reflection" velocity on the sphere. In all instances approximate expressions for the drag of the rigid spheres were given. Cunningham,⁷ Williams,⁸ and Lee⁹ considered the motion of a rigid sphere at the instant it passes the center of a spherical container. Bond¹⁰ suggested an approximate expression for the wall effect of fluid spheres in cylinders, based on Ladenburg's results.

In the present study, two problems are considered. The first problem deals with the steady, slow motion of rigid spheres along the axis of an infinitely long circular cylinder. The boundary conditions in this case are: uniform velocity on the surface of the sphere, zero velocity on the cylinder, zero velocity or parabolic velocity distribution at infinity. The solution of the problem is effected by means of the Stokes stream function for axisymmetric flow. The motion of the spheres in a cylinder is solved in terms of a system of linear algebraic equations for the constants in a series expansion for the stream function. Numerical results are obtained for the drag of the rigid spheres for diameter ratios up to 0.8. Values of wall correction factors have been computed for two special cases: namely, (1) when the sphere is in motion in a stationary liquid, and (2) when the liquid is in motion within the cylindrical tube past a stationary sphere. The drag of moving spheres within a moving liquid can be obtained by appropriate combination of these two correction factors. It is also shown that the first two equations of the infinite set closely approximate the drag over a large range of diameter ratios.

The second problem deals with the motion of fluid spheres (i.e., spheres which consist of a fluid that has different physical properties than the external medium, and are

*During publication of this report, a paper dealing with the wall effect of spheroids has come to the attention of the authors (Reference 30).

**The corresponding two-dimensional problem (a circular cylinder moving parallel with the two container walls) has been treated by Faxén³ and Tskulid⁴

¹References are listed on page 65.

characterized by internal motion) along the axis of a circular cylinder.* In contrast to the rigid case, the shape of fluid bodies cannot be specified beforehand. The shape is a consequence of the motion such that the boundary conditions are satisfied at the body interface. For the slow motion of a fluid body in an infinite medium, it was shown analytically by Hadamard¹² and Rybczynski¹³ that the sphere is a possible shape for which the boundary conditions (continuity of velocity and stress, no diffusion) are satisfied. Experimental evidence by Spells¹⁴ indicates that the spherical shape is actually attained by fluid bodies in slow motion. In the present analysis, a spherical shape is assumed for fluid bodies moving within a cylindrical tube, and an approximate solution (similar to the approximate solution for the rigid case) is obtained. As before, values of wall correction factors have been computed for the two special cases. The validity of the approximation is confirmed by experiments with drops moving in a still liquid. The experimental investigation shows that the shape of a fluid body in slow motion inside a cylinder is not exactly spherical. The drag of such nonspherical bodies can be evaluated from the theoretical solution if an equivalent radius based on its volume is used, and if the diameter ratio is smaller than 0.5. In general, it is shown that the wall effect for fluid spheres is less than for corresponding rigid spheres. Stream functions have been evaluated for rigid and fluid spheres in a stationary liquid for a diameter ratio of 0.5 and are presented in the form of streamlines about the spheres.

The investigations described in this report were carried out at the David Taylor Model Basin under NS 715-102 in connection with a program investigating gas-bubble dynamics. The study dealing with the motion of spheres in a still liquid was first presented in thesis form¹⁵ and subsequently in abbreviated form.¹⁶ The results pertaining to the motion of spheres within a moving liquid have been presented in abbreviated form.¹⁷

METHOD OF SOLUTION

The motions considered here are assumed to be sufficiently slow so that the Stokes equations of motion are an accurate description of the flow. For the steady, axisymmetric motion of an incompressible fluid, the Stokes stream function exists and satisfies a fourth-order differential equation. Corresponding to the two boundary shapes (cylinder and sphere), solutions for the stream function are obtained in cylindrical and spherical coordinates. In cylindrical coordinates, the differential equation for the stream function is:

*It is known that drops and bubbles below certain critical sizes depart from being "fluid" and become "rigid" in their behavior. (See, e.g., the authors' investigation on the motion of gas bubbles, Reference 11.) It is not the purpose of the present investigation to deal with this transition phenomenon nor to determine the conditions at which transition to "rigidity" takes place. It is intended to give the wall effect for those droplets or bubbles that behave as fluid bodies. For those behaving as rigid spheres, the wall effect for the rigid case applies.

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial}{\partial \rho} \right]^2 \Psi(x, \rho) = 0 \quad [1]$$

It will be shown that a solution of this equation (for the case of symmetry about the horizontal y - z plane) is:

$$\Psi(x, \rho) = \int_0^\infty [\rho K_1(a\rho) f(a) + \rho^2 K_0(a\rho) F(a) + \rho I_1(a\rho) g(a) + \rho^2 I_0(a\rho) G(a)] \cos ax da \quad [2]$$

In spherical coordinates, the stream function satisfies the differential equation

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1 - \cos^2 \theta}{r^2} \frac{\partial^2}{\partial (\cos \theta)^2} \right]^2 \Psi(r, \theta) = 0 \quad [3]$$

a solution of which is shown to be

$$\Psi(r, \theta) = \sum_{\substack{n=2 \\ n=\text{even}}}^{\infty} C_n^{-1/2}(\cos \theta) \left[A_n r^2 + B_n \frac{1}{r^{n-1}} + C_n r^{n+2} + D_n \frac{1}{r^{n-3}} \right] \quad [4]$$

where $x, \rho; r, \theta$ are coordinates,

$C_n^{-1/2}$ is Gegenbauer polynomial,* order n , degree $-1/2$,

A_n, B_n, C_n, D_n are constants,

I_0, I_1, K_0, K_1 are modified Bessel functions, and

$f(a), F(a), g(a), G(a)$ are arbitrary functions.

For the satisfaction of the boundary conditions on the cylinder walls, the cylindrical coordinate solution for the stream function is used. The expression thus obtained represents the flow inside a circular cylinder, not as yet fully specified but satisfying the boundary conditions on the cylinder. This expression is then transformed into spherical coordinates. By comparing termwise the constants in the above expression with the constants in the stream function expansion obtained directly in spherical coordinates, a relationship between the constants is obtained. The boundary conditions on the sphere yield a relationship between the constants in the spherical coordinate solution. Substituting the previous relationships into those obtained from the boundary conditions on the sphere, a set of linear algebraic equations for evaluating the constants is obtained.

*The first ten of these polynomials have been evaluated in Appendix A.

THE STREAM FUNCTION IN SPHERICAL COORDINATES

The Stokes stream function in spherical coordinates (Figure 1) satisfies the fourth-order differential equation (Reference 18, p. 393):

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1 - \cos^2 \theta}{r^2} \frac{\partial^2}{\partial (\cos \theta)^2} \right]^2 \Psi(r, \theta) = 0 \quad [3]$$

A solution of the differential equation is sought in the form

$$\Psi = \Psi_1 + \Psi_2 \quad [5]$$

such that $L \Psi_1 = 0$ and $L \Psi_2 = \Psi_1$ (i.e., $L^2 \Psi_2 = 0$)

where $L = \left[\frac{\partial^2}{\partial r^2} + \frac{1 - t^2}{r^2} \frac{\partial^2}{\partial t^2} \right]$ and

$$t = \cos \theta.$$

The solution is to be single-valued and continuous in the given flow field. Assume a solution for $L \Psi_1 = 0$ exists in product form; i.e.,

$$\Psi_1(r, \theta) = R(r) T(t) \quad [6]$$

The second-order partial differential equation can then be separated into two ordinary differential equations

$$r^2 R'' - a^2 R = 0 \quad [7]$$

$$(t^2 - 1) T'' - a^2 T = 0 \quad [8]$$

where the primes denote differentiation with respect to the argument.

The solution of Equation [7] is:

$$R(r) = c_1 r^m + c_2 r^{-m+1} \quad [9]$$

where $m = \frac{1}{2} + \frac{\sqrt{1+4a^2}}{2}$.

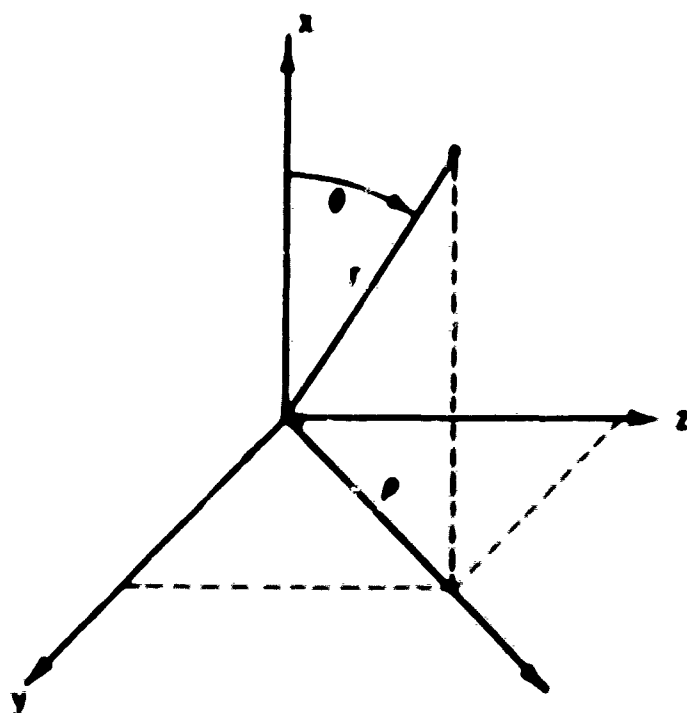


Figure 1 - Coordinate Systems

The solution of Equation (8) is:^{*}

$$T(t) = C_{\alpha}^{-1/2}(t) + c_3 F_{\alpha}(t) \quad (10)$$

where $C_{\alpha}^{-1/2}$ = Gegenbauer function of order α , degree $-1/2$, and

$$a^2 = \alpha(\alpha-1).$$

Comparing the expressions for m and α , we obtain $m = \alpha$. Hence

$$\Psi_{1\alpha}(r, \theta) = (c_1 r^{\alpha} + c_2 r^{-\alpha+1}) (C_{\alpha}^{-1/2} + c_3 F_{\alpha}) \quad (11)$$

The solution of the equation $L \Psi_2 = \Psi_1$ is

$$\Psi_2 = \Psi_1 + \Psi_p \quad (12)$$

Assume the particular solution Ψ_p in the form $S(r) T(t)$, hence

$$r^2 S'' - \alpha(\alpha-1) S = r^2 (c_1 r^{\alpha} + c_2 r^{-\alpha+1}) \quad (13)$$

and

$$S(r) = r^2 (c_4 r^{\alpha} + c_5 r^{-\alpha+1}) \quad (14)$$

Therefore

$$\Psi_{\alpha} = (C_{\alpha}^{-1/2} + c_3 F_{\alpha}) (c_1 r^{\alpha} + c_2 r^{-\alpha+1} + c_4 r^{-\alpha-2} + c_5 r^{-\alpha+3}) \quad (15)$$

Summing the above solution and using integer values of α , the expression for the stream function for axisymmetric, slow flow in spherical coordinates becomes:

$$\begin{aligned} \Psi(r, \theta) = & \sum_0^{\infty} C_n^{-1/2}(t) (A_n r^n + B_n r^{-n+1} + C_n r^{n+2} + D_n r^{-n+3}) + \\ & + \sum_2^{\infty} F_n(t) (A'_n r^n + B'_n r^{-n+1} + C'_n r^{n+2} + D'_n r^{-n+3}) \end{aligned} \quad (18)^{**}$$

^{*}The solution $C_{\alpha}^{-1/2} = \frac{1}{2\alpha-1} (P_{\alpha-2} - P_{\alpha})$ is given in Reference 19, p. 329; the second solution is $F_{\alpha} = \frac{1}{2\alpha-1}$

$(Q_{\alpha-2} - Q_{\alpha})$, where $Q_{\alpha}(t)$ = Legendre function of the second kind. $C_{\alpha}^{-1/2}$ has a singularity at $t = -1$ (i.e., $\theta = \pi$) except for $\alpha = \text{integer}$. F_{α} has singularities at $t = \pm 1$ (i.e., $\theta = 0, \pi$) for all values of α .

^{**}Šavic²⁰ gives a solution for the stream function in the form

$$\Psi = \sum_2^{\infty} C_n^{-1/2} (A_n r^n + B_n r^{-n+1} + C_n r^{n+2} + D_n r^{-n+3}),$$

The terms of the series expansion $\psi = \sum_{n=0}^{\infty} P_n(r) \zeta_n(\theta)$ and the P_n 's satisfy the second-order partial differential equation ($L\psi = 0$). The ζ_n and P_n terms satisfy the fourth-order partial differential equation ($L^2\psi = 0$). These constants are obtained from the specific boundary conditions of the problem.

From the relations

$$v_r = -\frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} \quad [17]$$

$$v_\theta = \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r}$$

expressions for the radial and tangential velocities of the flow field are obtained:

$$v_r = -\sum_{n=0}^{\infty} P_{n-1} (A_n r^{n-2} + B_n r^{n-1} + C_n r^n + D_n r^{n+1}) - \sum_{n=1}^{\infty} \zeta_{n-1} (A_n' r^{n-2} + B_n' r^{n-1} + C_n' r^n + D_n' r^{n+1}) \quad [18]$$

$$v_\theta = \frac{1}{r \sin \theta} \left[\sum_{n=0}^{\infty} \frac{P_n}{r} (A_n r^{n-1} + (n+1) B_n r^n + (n+2) C_n r^{n+1} + (n+3) D_n r^{n+2}) - \sum_{n=1}^{\infty} \frac{\zeta_n}{r \sin \theta} (A_n' r^{n-1} + (n-1) B_n' r^{n-2} + (n+1) C_n' r^{n-1} + (n+2) D_n' r^n) \right]$$

The pressure distribution is given by:

$$p = -\sum_{n=2}^{\infty} P_{n-1} \left[-\frac{2(2n+1)}{n-1} C_n r^{n-1} - \frac{2(2n-3)}{n} D_n r^n \right] + \mu \sum_{n=2}^{\infty} \zeta_{n-1} \left[-\frac{2(2n+1)}{n-1} C_n' r^{n-1} - \frac{2(2n-3)}{n} D_n' r^n \right] + \mu \left[-6D_0 C_0 - 6C_1 \ln(r \sin \theta) + 2D_1 \frac{1}{r} \right] \quad [19]$$

constant

The $C_n^{-1/2}(\theta)$ terms in the stream function expansion (Equation (16)) are not symmetrical about the $y-z$ plane for odd n 's; the $\zeta_n(\theta)$ terms are not symmetrical about the $y-z$ plane for even n 's. Furthermore, the polynomials $P_n(r)$ have singularities at $\theta = 0, \pi$. Also, the first two terms of the stream function expansion ($n = 0, 1$) result in infinite tangential velocities

The terms of the series expansion (15) are such that the B_n 's satisfy the second-order partial differential equation ($L^2 \Psi = 0$). The C_n and D_n terms satisfy the fourth-order partial differential equation ($L^2 \Psi = 0$). These constants are obtained from the specific boundary conditions of the problem.

From the relations

$$v_r = -\frac{1}{r^2 \sin \theta} \frac{\partial \Psi}{\partial \theta}$$

[17]

$$v_\theta = \frac{1}{r \sin \theta} \frac{\partial \Psi}{\partial r}$$

expressions for the radial and tangential velocities of the flow field are obtained:

$$v_r = -\sum_0^\infty P_{n-1} (A_n r^{n-2} + B_n r^{-n-1} + C_n r^n + D_n r^{-n+1})$$

$$- \sum_2^\infty Q_{n-1} (A'_n r^{n-2} + B'_n r^{-n-1} + C'_n r^n + D'_n r^{-n+1})$$

[18]

$$v_\theta = \sum_0^\infty \frac{P_{n-1}}{\sin \theta} [(n-1) A_n r^{n-1} - (n+1) B_n r^{-n} + (n+2) C_n r^{n+1} - (n+3) D_n r^{-n+2}]$$

$$+ \sum_2^\infty \frac{Q_{n-1}}{\sin \theta} [(n-1) A'_n r^{n-1} - (n+1) B'_n r^{-n} + (n+2) C'_n r^{n+1} - (n+3) D'_n r^{-n+2}]$$

The pressure distribution is given by:

$$p = \mu \sum_2^\infty P_{n-1} \left[-\frac{2(2n+1)}{n-1} C_n r^{n-1} - \frac{2(2n-3)}{n} D_n r^{-n} \right] + \mu \sum_2^\infty Q_{n-1}$$

$$\left[-\frac{2(2n+1)}{n-1} C'_n r^{n-1} - \frac{2(2n-3)}{n} D'_n r^{-n} \right] + \mu \left[-6 D_0 C_0 - 6 C_1 \ln(r \sin \theta) + 2 D_1 \frac{1}{r} \right] \quad [19]$$

+ constant

The $C_n^{-1/2}(t)$ terms in the stream function expansion (Equation (16)) are not symmetrical about the $y-z$ plane for odd n 's; the $B_n(t)$ terms are not symmetrical about the $y-z$ plane for even n 's. Furthermore, the polynomials $F_n(t)$ have singularities at $\theta = 0, \pi$. Also, the first two terms of the stream function expansion ($n = 0, 1$) result in infinite tangential velocity

for $n = 0, 1, 2, \dots$

$$n = 0$$

$$v_r = 0$$

$$v_\theta = \frac{1}{2} \left(\frac{R}{r} + 2C_1 r + 3C_2 r^2 \right)$$

$$n = 1$$

$$v_r = -\frac{A_1}{r} - \frac{B_1}{r^2} - C_1 r - D_1$$

$$v_\theta = -\cos \theta \left(\frac{A_1}{r} + 2C_1 r + 2D_1 \right)$$

THE STREAM FUNCTION IN CYLINDRICAL COORDINATES

A relation for the Stokes stream function in cylindrical coordinates is obtained in a manner similar to that used for spherical coordinates. The differential equation satisfied by the stream function is (see, for example, [1], p. 207)

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right) \psi(r, z) = 0 \quad (1)$$

With the assumption that the stream function is separable, i.e., it can be written as a product of two functions of r and z , respectively, we have

$$\psi(r, z) = R(r) Z(z) \quad (2)$$

Substituting (2) into (1) and dividing by RZ , we obtain

$$\frac{1}{R} \left(\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right) + \frac{d^2 Z}{dz^2} = 0 \quad (3)$$

Since the first term in (3) is a function of r only and the second term is a function of z only, we have

$$R'' + \frac{1}{r} R' + \lambda^2 R = 0 \quad (4)$$

$$Z'' - \lambda^2 Z = 0 \quad (5)$$

where λ^2 is an arbitrary constant.

$$Z(z) = C_1 \cosh \lambda z + C_2 \sinh \lambda z \quad (6)$$

where C_1 and C_2 are arbitrary constants. For $\lambda = 0$, see Reference [1], pp. 198, 199.

$$P(\rho) = c_3 I_1(\rho) + c_4 K_1(\rho) \quad [25]$$

where $I_1(\rho)$ is the modified Bessel function of the first kind, first order, and $K_1(\rho)$ is the modified Bessel function of the second kind, first order.

Hence

$$\Psi_{1a} = [c_3 \rho I_1(\rho) + c_4 \rho K_1(\rho)] (c_1 \cos az + c_2 \sin az) \quad [26]$$

For $a^2 = 0$, we obtain

$$\Psi_1 = (c_7 + c_8 \rho^2) (c_5 + c_6 z) \quad [27]$$

The particular solution of $L \Psi_2 = \Psi_1$ is sought in the form $Q(\rho) X(z)$; hence

$$Q'' - \frac{Q'}{\rho} + a^2 Q = \text{constant} \cdot P \quad [28]$$

A solution of Equation [28] is given by

$$\Psi_2 = \int_0^z \Psi_1 dz \quad [29]$$

where the integral denotes differentiation with respect to the argument. Therefore

$$\Psi_{2a} = [c_3 \rho^2 I_0(\rho) + c_4 \rho^2 K_0(\rho)] (c_1 \cos az + c_2 \sin az) \quad [30]$$

For $a^2 = 0$, we obtain

$$\Psi_2 = c_9 \rho^4 (c_5 + c_6 z) \quad [31]$$

A solution of Equation [1] is thus given by

$$\Psi_a = (c_3 \rho I_1 + c_4 \rho K_1 + c_5 \rho^2 I_0 + c_6 \rho^2 K_0) (c_1 \cos az + c_2 \sin az) \quad [32]$$

where I_0 , I_1 , K_0 , and K_1 are used in lieu of $I_0(\rho)$, $I_1(\rho)$, $K_0(\rho)$, and $K_1(\rho)$. Since the solution holds for all positive values of a , the integral form

$$\Psi = \int_0^\infty \Psi_a da \quad [33]$$

is also a solution of Equation [1].

$\Psi_2 = (c_{10} \rho^2 + c_{11} z + c_{12}) z^2$ is also a solution of $L \Psi_2 = \Psi_1$.

Thus letting $c_1 c_3 = g_1(a)$, $c_1 c_4 = f_1(a)$, $c_1 c_5 = G_1(a)$, $c_1 c_6 = F_1(a)$, $c_2 c_3 = g_2(a)$, $c_2 c_4 = f_2(a)$, $c_2 c_5 = G_2(a)$, and $c_2 c_6 = F_2(a)$, the expression for the stream function in cylindrical coordinates becomes:

$$\begin{aligned} \Psi(x, \rho) = & \int_0^\infty [\rho K_1 f_1(a) + \rho^2 K_0 F_1(a) + \rho I_1 g_1(a) + \rho^2 I_0 G_1(a)] \cos ax da \\ & + \int_0^\infty [\rho K_1 f_2(a) + \rho^2 K_0 F_2(a) + \rho I_1 g_2(a) + \rho^2 I_0 G_2(a)] \sin ax da \quad [34] \\ & + k_1 \rho^2 + k_2 \rho^4 + k_3 \rho^2 x + k_4 \rho^4 x + k_5 x^2 + k_6 \rho^2 x^2 + k_7 x^3 + k_8 x \end{aligned}$$

The terms containing $f(a)$ and $g(a)$ satisfy the second-order differential equation ($L \Psi = 0$); the $F(a)$ and $G(a)$ terms satisfy the fourth-order differential equation ($L^2 \Psi = 0$). The arbitrary functions $f(a)$, $F(a)$, $g(a)$, and $G(a)$ are evaluated from the boundary conditions of the problem.

From the relations

$$\begin{aligned} v_x &= -\frac{1}{\rho} \frac{\partial \Psi}{\partial \rho} \\ v_\rho &= \frac{1}{\rho} \frac{\partial \Psi}{\partial x} \end{aligned} \quad [35]$$

the velocities in the axial and ρ -direction are obtained:

$$\begin{aligned} v_x = & \int_0^\infty [a K_0 f_1(a) - (2K_0 - a\rho K_1) F_1(a) - a I_0 g_1(a) - (2I_0 + a\rho I_1) G_1(a)] \cos ax da \\ & + \int_0^\infty [a K_0 f_2(a) - (2K_0 - a\rho K_1) F_2(a) - a I_0 g_2(a) - (2I_0 + a\rho I_1) G_2(a)] \sin ax da \\ & - 2k_1 - 4k_2 \rho^2 - 2k_3 x - 4k_4 \rho^2 x - 2k_5 x^2 \\ & v_\rho = - \int_0^\infty [K_1 f_1(a) + \rho K_0 F_1(a) + I_1 g_1(a) + \rho I_0 G_1(a)] a \sin ax da \\ & + \int_0^\infty [K_1 f_2(a) + \rho K_0 F_2(a) + I_1 g_2(a) + \rho I_0 G_2(a)] a \cos ax da \\ & + k_3 \rho + k_4 \rho^3 + 2k_5 \frac{x}{\rho} + 2k_6 \rho x + k_7 \frac{x^2}{\rho} + k_8 \frac{1}{\rho} \end{aligned} \quad [36]$$

The expression for the pressure field is:

$$\begin{aligned}
 p = & -2\mu \int_0^\infty [I_0 G_1(a) + K_0 F_1(a)] a \sin az da \\
 & + 2\mu \int_0^\infty [I_0 G_2(a) + K_0 F_2(a)] a \cos az da + \mu (-16k_2 x - 8k_4 x^2 + 4k_4 \rho^2 - 4k_6 x + 6k_7 \ln \rho) \\
 & + \text{constant}
 \end{aligned}
 \tag{37}$$

THE MOTION OF SPHERES IN A CYLINDRICAL TUBE

A. MOTION IN A STATIONARY LIQUID

1. Rigid Spheres

a. **Exact Solution.** The first problem considered is the motion of a sphere within a still liquid of finite extent, as occurs when a sphere rises or falls under the influence of gravity. In this section the axial motion of a rigid sphere inside an infinitely long cylinder will be dealt with.*

For convenience, the coordinate origin is taken at the center of the sphere and the cylinder is assumed to be moving at constant velocity U in the negative x -direction (Figure 2). The boundary conditions are:

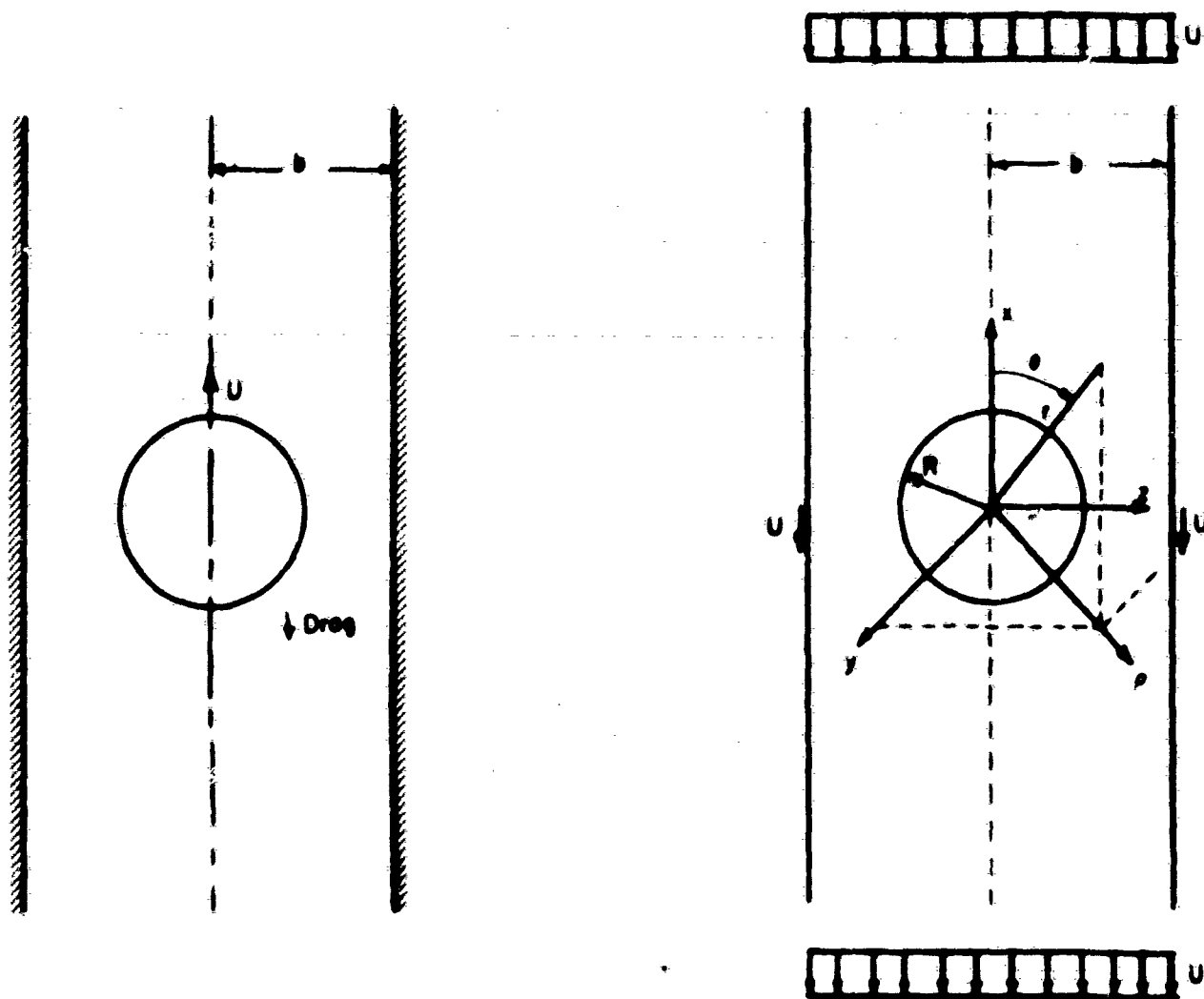
$$\begin{aligned}
 \text{at the cylinder walls } (\rho = b): \quad & v_x = -U; \quad v_\rho = 0 \quad \text{or} \quad \Psi = \frac{U}{2} b^2 \\
 \text{at infinity } (x = \pm \infty): \quad & v_x = -U; \quad v_\rho = 0 \\
 \text{on the surface of the sphere } (r = R): \quad & v_r = 0; \quad v_\theta = 0
 \end{aligned}
 \tag{38}$$

The case of a sphere moving at constant velocity U is obtained from the above by superimposing a uniform flow with velocity U in the positive x -direction.

Since the stream function is symmetrical about the y - z plane because of symmetry in the boundary conditions, the stream function in spherical coordinates is given by

$$\Psi(r, \theta) = \sum_{n=2}^{\infty} C_n^{-1/2} (\cos \theta) \left(A_n r^n + B_n \frac{1}{r^{n-1}} + C_n r^{n+2} + D_n \frac{1}{r^{n-3}} \right) \quad (n = 2, 4, 6, \dots)
 \tag{39}$$

*The case of the motion of spheres at the center of a spherical container (neglecting any unsteady effects) is considered in Appendix E.



Sphere Moving at Velocity U in Still Liquid
Inside Fixed Cylindrical Tube.

Equivalent System: Sphere Fixed; Tube Moving
at Velocity U . Uniform Velocity U at Infinity.

Figure 2 - Definition Sketch

The boundary conditions on the cylinder walls can be satisfied if we take the stream function in cylindrical coordinates as

$$\Psi(x, \rho) = \int_0^\infty [\rho K_1(a\rho) f_1(a) + \rho^2 K_0(a\rho) F_1(a) + \rho I_1(a\rho) g_1(a) + \rho^2 I_0(a\rho) G_1(a)] \cos ax da + \frac{U}{2} \rho^2 \quad [40]^*$$

Utilizing the boundary conditions on the cylinder walls (Equation [38]), we obtain the following relations:

$$\text{from } \Psi = \frac{U}{2} b^2: b K_1(ab) f_1(a) + b^2 K_0(ab) F_1(a) + b I_1(ab) g_1(a) + b^2 I_0(ab) G_1(a) = 0$$

$$\begin{aligned} \text{from } v_x = -U: & -a K_0(ab) f_1(a) + [2 K_0(ab) - a b K_1(ab)] F_1(a) + a I_0(ab) g_1(a) \\ & + [2 I_0(ab) + a b I_1(ab)] G_1(a) = 0 \end{aligned} \quad [41]$$

*The integral containing " $\sin ax$ " is not symmetric with respect to the $y-z$ plane.

We can, therefore, solve for $g_1(a)$ and $G_1(a)$ in terms of $f_1(a)$ and $F_1(a)$

$$g_1(a) = - \frac{I_1(ab)K_1(ab) + I_0(ab)K_2(ab)}{I_1^2(ab) - I_0(ab)I_2(ab)} f_1(a) - \frac{1}{a} \frac{1}{I_1^2(ab) - I_0(ab)I_2(ab)} F_1(a) \quad [42]$$

$$G_1(a) = \frac{a}{(ab)^2} \frac{1}{I_1^2(ab) - I_0(ab)I_2(ab)} f_1(a) + \frac{I_1(ab)K_1(ab) + I_2(ab)K_0(ab)}{I_1^2(ab) - I_0(ab)I_2(ab)} F_1(a)$$

where the following relations for the Bessel functions have been used:

$$I_0(ab)K_1(ab) + I_1(ab)K_0(ab) = \frac{1}{ab} \quad \text{and} \quad I_2(ab) = I_0(ab) - \frac{2}{ab} I_1(ab)$$

Let

$$\begin{aligned} S_1 &= - \frac{I_1(ab)K_1(ab) + I_0(ab)K_2(ab)}{[I_1(ab)]^2 - I_0(ab)I_2(ab)} \\ S_2 &= \frac{a}{(ab)^2} \frac{1}{[I_1(ab)]^2 - I_0(ab)I_2(ab)} \\ S_3 &= - \frac{1}{a} \frac{1}{[I_1(ab)]^2 - I_0(ab)I_2(ab)} \\ S_4 &= \frac{I_1(ab)K_1(ab) + I_2(ab)K_0(ab)}{[I_1(ab)]^2 - I_0(ab)I_2(ab)} \end{aligned} \quad [43]$$

Utilizing $I_0(ab)K_2(ab) - I_2(ab)K_0(ab) = \frac{2}{(ab)^2}$, we obtain

$$g_1(a) = \left[\frac{2a}{(ab)^2} S_3 - S_4 \right] f_1(a) + S_3 F_1(a) \quad [44]$$

$$G_1(a) = S_2 f_1(a) + S_4 F_1(a)$$

Thus having satisfied the boundary conditions on the cylinder walls, the stream function becomes

$$\Psi(x, \rho) = \int_0^\infty \{ \rho K_1(a\rho) f_1(a) + \rho^2 K_0(a\rho) F_1(a) + \rho I_1(a\rho) [S_1 f(a) + S_3 F_1(a)] + \\ + \rho^2 I_0(a\rho) [S_2 f_1(a) + S_4 F_1(a)] \} \cos ax da + \frac{U\rho^2}{2} \quad [45]$$

In order to transform the above expression into a form amenable to the satisfaction of the boundary conditions on the sphere, let

$$f_1(a) = \sum_1^\infty a_n a^n \quad F_1(a) = \sum_0^\infty b_n a^n \quad [46]$$

The integrals $\rho^2 \int_0^\infty K_0(a\rho) a^n \cos ax da$ and $\rho \int_0^\infty K_1(a\rho) a^n \cos ax da$ which then result are evaluated in Appendix B. It is shown there that only odd powers of a for a_n and even powers of a for b_n are of interest here; otherwise discontinuities result at $\theta = 0, \pi$. Thus we obtain from Equation [40]

$$\Psi(x, \rho) = \int_0^\infty \rho K_1(a\rho) (a_1 a + a_3 a^3 + a_5 a^5 + \dots) \cos ax da \\ + \int_0^\infty \rho^2 K_0(a\rho) (b_0 + b_2 a^2 + b_4 a^4 + \dots) \cos ax da \quad [47] \\ + \int_0^\infty [\rho I_1(a\rho) g_1(a) + \rho^2 I_0(a\rho) G_1(a)] \cos ax da + \frac{U\rho^2}{2}$$

The integrals containing I_0 and I_1 can be expanded into a Taylor series about the origin. The convergence of these integrals is assured by the form of the functions $g_1(a)$ and $G_1(a)$.

A function $\phi(x, \rho)$ can be expanded by Taylor's theorem into

$$\phi(x, \rho) = \phi(0, 0) + x \phi_x(0, 0) + \rho \phi_\rho(0, 0) \\ + \frac{1}{2!} [x^2 \phi_{xx}(0, 0) + 2x\rho \phi_{x\rho}(0, 0) + \rho^2 \phi_{\rho\rho}(0, 0)] + \dots \\ \dots + \frac{1}{n!} [x^n \phi_{x^n}(0, 0) + \binom{n}{1} x^{n-1} \rho \phi_{x^{n-1}\rho}(0, 0) + \dots \\ + \binom{n}{r} x^{n-r} \rho^r \phi_{x^{n-r}\rho^r}(0, 0) + \dots + \rho^n \phi_{\rho^n}(0, 0)] + \dots$$

where $\binom{n}{r}$ are binominal coefficients.

*The boundary conditions at infinity are satisfied, since the integrals $\int_0^\infty \left\{ \frac{\sin ax}{\cos ax} \right\} da$ vanish for $x \rightarrow \infty$.

Let

$$\phi(x, \rho) = \int_0^\infty [\rho I_1(a\rho) g_1(a) + \rho^2 I_0(a\rho) G_1(a)] \cos ax \, da \quad [48]$$

Hence

$$\phi_{x^n}(0, 0) = 0$$

$$\phi_{\rho^n}(0, 0) = 0 \quad n = \text{odd}$$

$$\phi_{\rho\rho}(0, 0) = \int_0^\infty [g_1(a)a + 2G_1(a)] \, da$$

$$\phi_{\rho^4}(0, 0) = \int_0^\infty [3/2 g_1(a) a^3 + 6G_1(a) a^2] \, da$$

$$\phi_{\rho^6}(0, 0) = \int_0^\infty [15/8 g_1(a) a^5 + 15G_1(a) a^4] \, da$$

etc.

$$\phi_{x^n x^n}(0, 0) = 0 \quad n = \text{odd}$$

$$= (-1)^{n/2} \int_0^\infty (\text{same as in } \phi_{\rho^n}) a^n \, da \quad n = \text{even}$$

Let

$$\begin{aligned} a_2 &= \int_0^\infty [g_1(a)a + 2G_1(a)] \, da = - \int_0^\infty S_4 a f_1(a) \, da + \int_0^\infty (S_3 a + 2S_4) F_1(a) \, da \\ a_4 &= \int_0^\infty [g_1(a)a + 2G_1(a)] a^2 \, da = - \int_0^\infty S_4 a^3 f_1(a) \, da + \int_0^\infty (S_3 a + 2S_4) a^2 F_1(a) \, da \\ a_6 &= \int_0^\infty [g_1(a)a + 2G_1(a)] a^4 \, da = - \int_0^\infty S_4 a^5 f_1(a) \, da + \int_0^\infty (S_3 a + 2S_4) a^4 F_1(a) \, da \\ b_2 &= \int_0^\infty G_1(a) a^2 \, da = \int_0^\infty S_2 a^2 f_1(a) \, da + \int_0^\infty S_4 a^2 F_1(a) \, da \\ b_4 &= \int_0^\infty G_1(a) a^4 \, da = \int_0^\infty S_2 a^4 f_1(a) \, da + \int_0^\infty S_4 a^4 F_1(a) \, da \end{aligned} \quad [49]$$

Hence

$$\begin{aligned}\phi(x, \rho) = & \frac{1}{2} \rho^2 \alpha_2 + \frac{1}{4} x^2 \rho^2 (-\alpha_4) + \frac{1}{24} \rho^4 \left(\frac{3}{2} \alpha_4 + 3 \beta_4 \right) \\ & + \frac{1}{48} x^2 \rho^4 \left(-\frac{3}{2} \alpha_6 - 3 \beta_6 \right) + \frac{1}{48} x^4 \rho^2 \alpha_6 + \frac{1}{720} \rho^6 \left(\frac{15}{8} \alpha_6 + \frac{15}{2} \beta_6 \right) \\ & + \frac{1}{1440} x^2 \rho^6 \left(-\frac{15}{8} \alpha_8 - \frac{15}{2} \beta_8 \right) + \frac{1}{576} x^4 \rho^4 \left(\frac{3}{2} \alpha_8 + 3 \beta_8 \right) + \frac{1}{1440} x^6 \rho^2 (-\alpha_8) + \dots\end{aligned}\quad [50]$$

Since $f_1(a) = a_1 a + a_3 a^3 + a_5 a^5 + \dots$ and $F_1(a) = b_0 + b_2 a^2 + b_4 a^4 + \dots$, Equation [47] is thus transformed into

$$\begin{aligned}\Psi = & a_1 \frac{\pi}{r} C_2^{-1/2} - a_3 \frac{12\pi}{r^3} C_4^{-1/2} + a_5 \frac{360\pi}{r^5} C_6^{-1/2} - \dots + b_0 \pi r C_2^{-1/2} + b_2 \frac{2\pi}{5r} C_2^{-1/2} - b_2 \frac{12\pi}{5r} C_4^{-1/2} - \\ & - b_4 \frac{16\pi}{r^3} C_4^{-1/2} + b_4 \frac{40\pi}{r^3} C_6^{-1/2} + \dots \\ & + \frac{1}{2} \rho^2 \left[- \int_0^\infty S_4 (a_1 a^2 + a_3 a^4 + \dots) da + \int_0^\infty (S_3 a + 2 S_4) (b_0 + b_2 a^2 + \dots) da + U \right] + \\ & + \frac{1}{4} x^2 \rho^2 \left[\int_0^\infty S_4 (a_1 a^4 + a_3 a^6 + \dots) da - \int_0^\infty (S_3 a + 2 S_4) (b_0 a^2 + b_2 a^4 + \dots) da \right] + \\ & + \frac{1}{24} \rho^4 \left[\int_0^\infty \left(3 S_2 - \frac{3}{2} S_4 a \right) (a_1 a^3 + a_3 a^5 + \dots) da + \int_0^\infty \left(\frac{3}{2} S_3 a + 6 S_4 \right) (b_0 a^2 + b_2 a^4 + \dots) da \right] + \\ & + \frac{1}{48} x^2 \rho^2 \left[\int_0^\infty \left(-3 S_2 + \frac{3}{2} S_4 a \right) (a_1 a^5 + a_3 a^7 + \dots) da - \int_0^\infty \left(\frac{3}{2} S_3 a + 6 S_4 \right) (b_0 a^4 + b_2 a^6 + \dots) da \right] + \dots\end{aligned}\quad [51]$$

From the solution in spherical coordinates (Equation [39]), we obtain (making use of the relations $\rho/r = \sin \theta$ and $x/r = \cos \theta$):

$$\begin{aligned}\Psi = & C_2^{-1/2} \left(B_2 \frac{1}{r} + D_2 r \right) + \frac{1}{2} \rho^2 A_2 + x^2 \rho^2 \left(\frac{1}{2} A_4 + \frac{1}{2} C_2 \right) + \rho^4 \left(-\frac{1}{8} A_4 + \frac{1}{2} C_2 \right) \\ & + C_4^{-1/2} \left(B_4 \frac{1}{r^3} + D_4 \frac{1}{r} \right) + x^2 \rho^4 \left(-\frac{3}{4} A_6 + \frac{3}{8} C_4 \right) + x^4 \rho^2 \left(\frac{1}{2} A_6 + \frac{1}{2} C_4 \right) + \rho^6 \left(\frac{1}{16} A_6 - \frac{1}{8} C_4 \right) \\ & + C_6^{-1/2} \left(B_6 \frac{1}{r^5} + D_6 \frac{1}{r^3} \right) + x^2 \rho^6 \left(\frac{15}{16} A_8 - \frac{11}{16} C_6 \right) + x^2 \rho^4 \left(-\frac{15}{8} A_8 - \frac{1}{4} C_6 \right) \\ & + x^6 \rho^2 \left(\frac{1}{2} A_8 + \frac{1}{2} C_6 \right) + \rho^8 \left(-\frac{5}{128} A_8 - \frac{1}{16} C_6 \right) + \dots\end{aligned}\quad [52]$$

By comparing Equations [51] and [52], the following relationships are obtained between the constants A_n , B_n , C_n , D_n and a_n , b_n :

$$B_2 = \left(a_1 + \frac{2}{5} b_2 \right) \pi$$

$$B_4 = -(12 a_3 + 16 b_4) \pi$$

$$B_n = (-1)^{\frac{n}{2}+1} \left[\frac{n!}{2} a_{n-1} + \frac{n(n-1)n!}{2(2n+1)} b_n \right] \pi$$

$$D_2 = b_0 \pi$$

$$D_4 = -\frac{12}{5} b_2 \pi$$

$$D_n = (-1)^{\frac{n}{2}+1} \frac{n!}{2(2n-3)} b_{n-2} \pi \quad [53a]$$

$$A_2 = a_2 + U$$

$$A_4 = -\frac{a_4}{2} - \frac{\beta_4}{5}$$

$$A_n = (-1)^{\frac{n}{2}+1} \left[\frac{a_n}{(n-2)!} + \frac{\beta_n}{(n-4)!(2n-3)} \right]$$

$$C_2 = \frac{\beta_4}{5}$$

$$C_n = (-1)^{\frac{n}{2}+1} \frac{\beta_{n+2}}{(n-2)!(2n+1)}$$

To facilitate the evaluation of a_n and β_n , let

$$S_2^n = \int_0^\infty S_2(ab)^n b d(ab)$$

$$S_3^n = \int_0^\infty S_3(ab)^n \frac{d(ab)}{b}$$

$$S_4^n = \int_0^\infty S_4(ab)^n d(ab)$$

This results in

$$\alpha_n = -a_1 \frac{S_4^n}{b^{n+1}} - a_3 \frac{S_4^{n+2}}{b^{n+3}} - \dots - a_m \frac{S_4^{n+m-1}}{b^{n+m}} - \dots$$

$$+ b_0 \frac{1}{b^{n-1}} (S_3^{n-1} + 2S_4^{n-2}) + b_2 \frac{1}{b^{n+1}} (S_3^{n+1} + 2S_4^n) + \dots + b_m \frac{1}{b^{n+m-1}} (S_3^{n+m-1} + 2S_4^{n+m-2}) + \dots$$

$$\beta_n = a_1 \frac{S_2^{n-1}}{b^{n+1}} + a_3 \frac{S_2^{n+1}}{b^{n+3}} + \dots + a_m \frac{S_2^{n+m-2}}{b^{n+m}} + \dots$$

$$+ b_0 \frac{S_4^{n-2}}{b^{n-1}} + b_2 \frac{S_4^n}{b^{n+1}} + \dots + b_m \frac{S_4^{n+m-2}}{b^{n+m-1}} + \dots$$

A number of the S_2^n and S_4^n integrals have been evaluated in Appendix C and are tabulated in Table 3. Thus we obtain

$$A_2 = -\frac{6.55507\lambda^3}{R^3} a_1 - \frac{15.3046\lambda^5}{R^5} a_3 - \dots - \frac{4.40866\lambda}{R} b_0 - \frac{4.76313\lambda^3}{R^3} b_2 - \dots + U$$

$$A_4 = \frac{4.07765}{R^5} \lambda^5 a_1 + \frac{54.7950}{R^7} \lambda^7 a_3 + \dots + \frac{1.07056}{R^3} \lambda^3 b_0 + \frac{11.9275}{R^5} \lambda^5 b_2 + \dots$$

$$A_n = (-1)^{\frac{n}{2}+1} \frac{1}{R^{n-2}} \left\{ \sum_{m=0,2,\dots}^{\infty} \left[\frac{1}{(n-2)!} S_3^{n+m-1} + \frac{n(n-1)}{(n-2)!(2n-3)} S_4^{n+m-2} \right] \frac{b_m}{R^{m+1}} \lambda^{n+m-1} + \right.$$

$$\left. + \sum_{m=1,3,\dots}^{\infty} \left[-\frac{1}{(n-2)!} S_4^{n+m-1} + \frac{1}{(n-4)!(2n-3)} S_2^{n+m-2} \right] \frac{a_m}{R^{m+2}} \lambda^{n+m} \right\} \quad [58b]$$

$$C_2 = \frac{2.1045\lambda^5}{R^3} a_1 + \frac{12.1172\lambda^7}{R^7} a_3 + \dots + \frac{1.81101\lambda^3}{R^3} b_0 + \frac{3.08092\lambda^5}{R^5} b_2 + \dots$$

$$C_n = (-1)^{\frac{n}{2}+1} \frac{1}{R^n} \frac{1}{(n-2)!(2n-1)} \left[\sum_{m=1,3,\dots}^{\infty} S_2^{n+m} \frac{a_m}{R^{m+2}} \lambda^{n+m+2} + \sum_{m=0,2,\dots}^{\infty} S_4^{n+m} \frac{b_m}{R^{m+1}} \lambda^{n+m+1} \right]$$

wh. $\lambda = \frac{\text{diameter of sphere}}{\text{diameter of cylinder}}$.

From the boundary conditions on the sphere (Equation [38]) and the orthogonality relations of Gegenbauer and Legendre polynomials,* we obtain

$$\text{from } v_r = 0: \quad A_n R^{n-2} + B_n \frac{1}{R^{n+1}} + C_n R^n + D_n \frac{1}{R^{n-1}} = 0 \quad [54]$$

$$\text{from } v_\theta = 0: \quad n A_n R^{n-2} - (n-1) B_n \frac{1}{R^{n+1}} + (n+2) C_n R^n - (n-3) D_n \frac{1}{R^{n-1}} = 0$$

Solving for A_n and C_n , we obtain

$$\begin{aligned} A_n &= -\frac{2n+1}{2} B_n \frac{1}{R^{2n-1}} - \frac{2n-1}{2} D_n \frac{1}{R^{2n-3}} \\ C_n &= \frac{2n-1}{2} B_n \frac{1}{R^{2n+1}} + \frac{2n-3}{2} D_n \frac{1}{R^{2n-1}} \end{aligned} \quad [55]$$

Substituting the expressions for A_n , B_n , C_n , D_n as given in Equations [53] into [55], an infinite set of linear algebraic equations is obtained for determining the constants a_1, a_3, \dots and b_0, b_2, \dots :

from A_2 :

$$\frac{b_0}{R} \left(\frac{3}{2} \pi - 4.40866 \lambda \right) + \frac{b_2}{R^3} (\pi - 4.76813 \lambda^3) + \dots + \frac{a_1}{R^3} \left(\frac{5}{2} \pi - 6.55507 \lambda^3 \right) + \frac{a_3}{R^5} (-15.8046 \lambda^5) + \dots = -U$$

from C_2 :

$$\frac{b_0}{R} \left(\frac{5}{2} \pi - 6.55507 \lambda^3 \right) + \frac{b_2}{R^3} (3\pi - 15.8046 \lambda^5) + \dots + \frac{a_1}{R^3} \left(\frac{15}{2} \pi - 17.8733 \lambda^5 \right) + \frac{a_3}{R^5} (-60.5861 \lambda^7) + \dots = 0$$

from A_4 :

$$\frac{b_0}{R} (-1.07056 \lambda^3) + \frac{b_2}{R^3} \left(\frac{42}{5} \pi - 11.9273 \lambda^5 \right) + \dots + \frac{a_1}{R^3} (-4.07763 \lambda^5) + \frac{a_3}{R^5} (54\pi - 54.7950 \lambda^7) + \dots = 0 \quad [56]$$

from C_4 :

$$\frac{b_0}{R} (-0.84533 \lambda^5) + \frac{b_2}{R^3} (6\pi - 7.43469 \lambda^7) + \dots + \frac{a_1}{R^3} (-8.36589 \lambda^7) + \frac{a_3}{R^5} (42\pi - 37.5381 \lambda^9) + \dots = 0$$

$$\int_{-1}^1 x^{-n} (1-x^2)^{-1/2} dx = 0 \quad n \neq 0 \quad = \frac{\Gamma(n-1)}{(n-1/2)n!} \quad n = 1$$

$$\int_{-1}^1 P_n(x) P_m(x) dx = 0 \quad m \neq n \quad = \frac{2}{2n+1} \quad m = n$$

from A_n :

$$\begin{aligned} \frac{(2n-1)n!}{4(2n-3)} \frac{b_{n-2}}{R^{n-1}} \pi + \frac{n(n-1)n!}{4} \frac{b_n}{R^{n+1}} \pi + \sum_{m=0,2,\dots}^{\infty} \left[\frac{1}{(n-2)!} S_3^{n+m-1} + \frac{n(n-1)}{(n-2)!(2n-3)} S_4^{n+m-2} \right] \\ \frac{b_m}{R^{m+1}} \lambda^{n+m-1} + \frac{(2n+1)n!}{4} \frac{a_{n-1}}{R^{n+1}} \pi + \sum_{m=1,3,\dots}^{\infty} \left[-\frac{1}{(n-2)!} S_4^{n+m-1} + \frac{1}{(n-4)!(2n-3)} S_2^{n+m-2} \right] \\ \frac{a_m}{R^{m+2}} \lambda^{n+m} = 0 \end{aligned} \quad [56]$$

from C_n :

$$\begin{aligned} \frac{n!}{4} \frac{b_{n-2}}{R^{n-1}} \pi + \frac{(2n-1)n(n-1)n!}{4(2n+1)} \frac{b_n}{R^{n+1}} \pi - \sum_{m=0,2,\dots}^{\infty} \frac{b_m}{R^{m+1}} \frac{S_4^{n+m}}{(n-2)!(2n+1)} \lambda^{n+m+1} \\ + \frac{(2n-1)n!}{4} \frac{a_{n-1}}{R^{n+1}} \pi - \sum_{m=1,3,\dots}^{\infty} \frac{a_m}{R^{m+2}} \frac{S_2^{n+m}}{(n-2)!(2n+1)} \lambda^{n+m+2} = 0 \end{aligned}$$

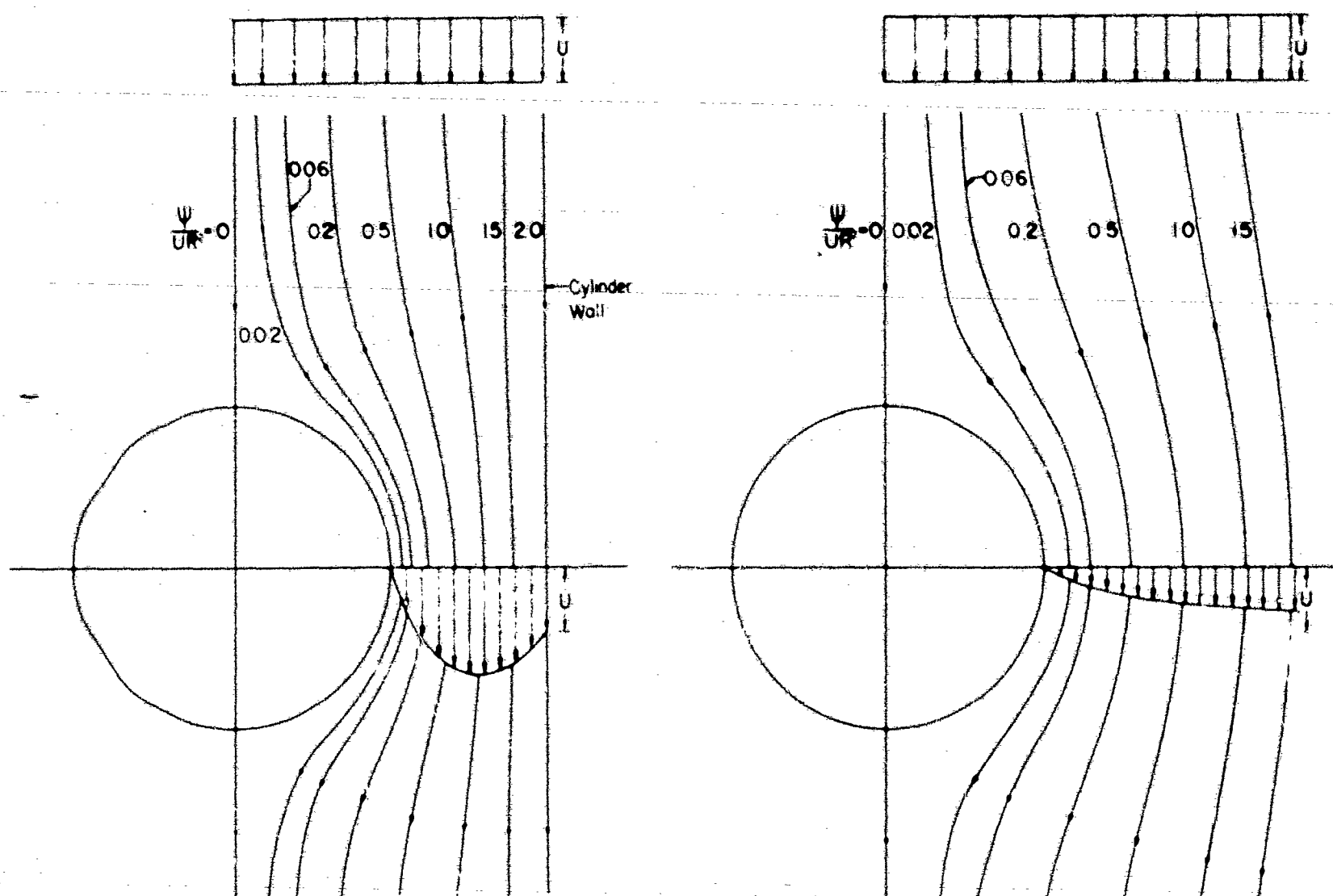
The constants a_n and b_n can be evaluated numerically to any degree of accuracy by increasing the number of equations utilized.

The solution for the stream function in spherical coordinates for flow about a sphere in an infinitely long cylinder is thus given by Equation [39] with the constants A_n , B_n , C_n , and D_n given in Equations [53]. Using ten equations of the algebraic set (Equations [56]), a sufficient number of the constants a_n and b_n was determined for a diameter ratio of 0.5. Streamlines were then evaluated from Equation [39]. They are shown in Figures 3 and 4. Figure 3 shows the relative motion about the sphere; i.e., the streamlines as they appear to an observer moving with the sphere. Figure 4 shows the absolute motion about the sphere; i.e., the streamlines as they appear to an observer fixed in space. Velocity distributions at the plane of symmetry and streamlines for the motion of a corresponding sphere in an infinite medium have been included in both figures.*

*The expression for the stream function for the motion of rigid spheres in an infinite medium is:

$$\Psi(r, \theta) = \frac{1}{2} \sin^2 \theta \left(U r^2 + \frac{1}{2} U R^3 \frac{1}{r} - \frac{3}{2} U R r \right) \quad \text{relative motion}$$

$$\Psi(r, \theta) = \frac{1}{2} \sin^2 \theta \left(\frac{1}{2} U R^3 \frac{1}{r} - \frac{3}{2} U R r \right) \quad \text{absolute motion}$$



Sphere in Circular Cylinder, Diameter Ratio: 0.5

Sphere in Infinite Medium

Figure 3 – Streamlines and Velocity Distribution for Rigid Spheres
(Exact Solution); Relative Motion

The drag experienced by moving spheres can be evaluated by integrating the forces acting over the surface of the sphere. Thus

$$\text{Drag} = \underbrace{\int p_{r\theta} \sin \theta dS}_{\text{Tangential Drag}} - \underbrace{\int p_{rr} \cos \theta d\theta}_{\text{Normal Drag}} \quad [57]$$

where $p_{r\theta}$ is the tangential stress,

p_{rr} is the normal stress, and

dS is the surface element.

The drag of the sphere can be obtained from the stream function and is shown in Appendix D to be equal to

$$\text{Drag} = -4\pi\mu D_2 = -4\pi^2\mu b_0 \quad [58]$$

i.e., the drag is proportional to the coefficient D_2 of the stream function expansion. Defining a wall correction factor (K_1) as the ratio of the drag of the sphere in the bounded medium to

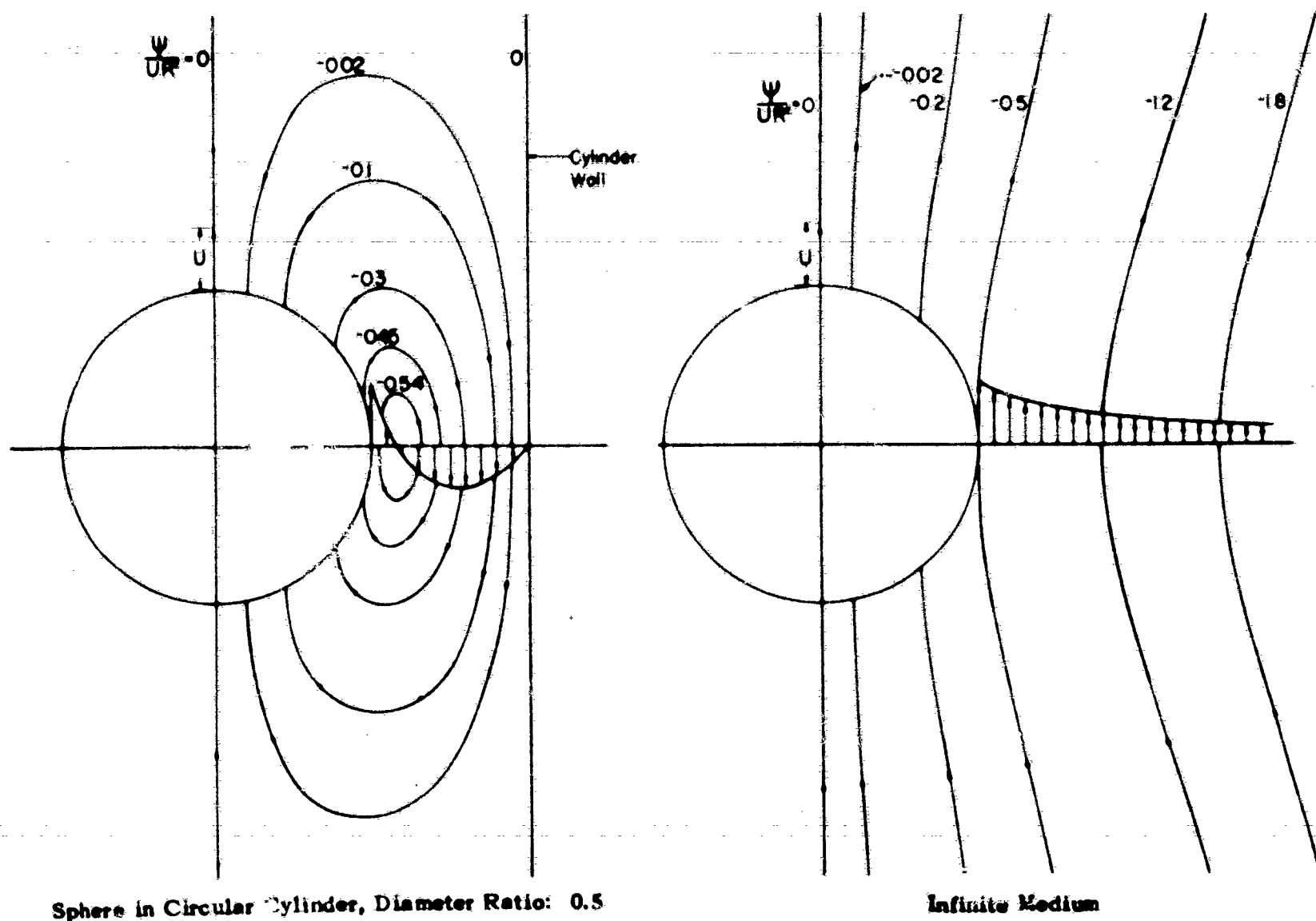


Figure 4 – Streamlines and Velocity Distribution for Rigid Spheres
(Exact Solution); Absolute Motion

that in an infinite medium, the drag can be expressed as

$$\text{Drag} = \underbrace{6\pi\mu UR}_{\text{Drag in Infinite Medium}} K_1 \quad [59]$$

Also, combining Equations [58] and [59], we obtain an expression for the wall correction factor in terms of the coefficient b_0 :

$$K_1 = -\frac{4s^2\mu b_0}{6\pi\mu UR} = -\frac{2\pi}{3} \frac{b_0}{UR} \quad [60]$$

Wall correction factors for rigid spheres moving in a still liquid inside an infinitely long cylinder have been determined by numerically solving the algebraic system (Equation [56]) for the coefficient b_0 over a large range of diameter ratios. The number of equations of the algebraic system used was increased (at most up to eight) until only very small changes in the value of b_0 were obtained. The computed wall correction factors are shown in Figure 5 and

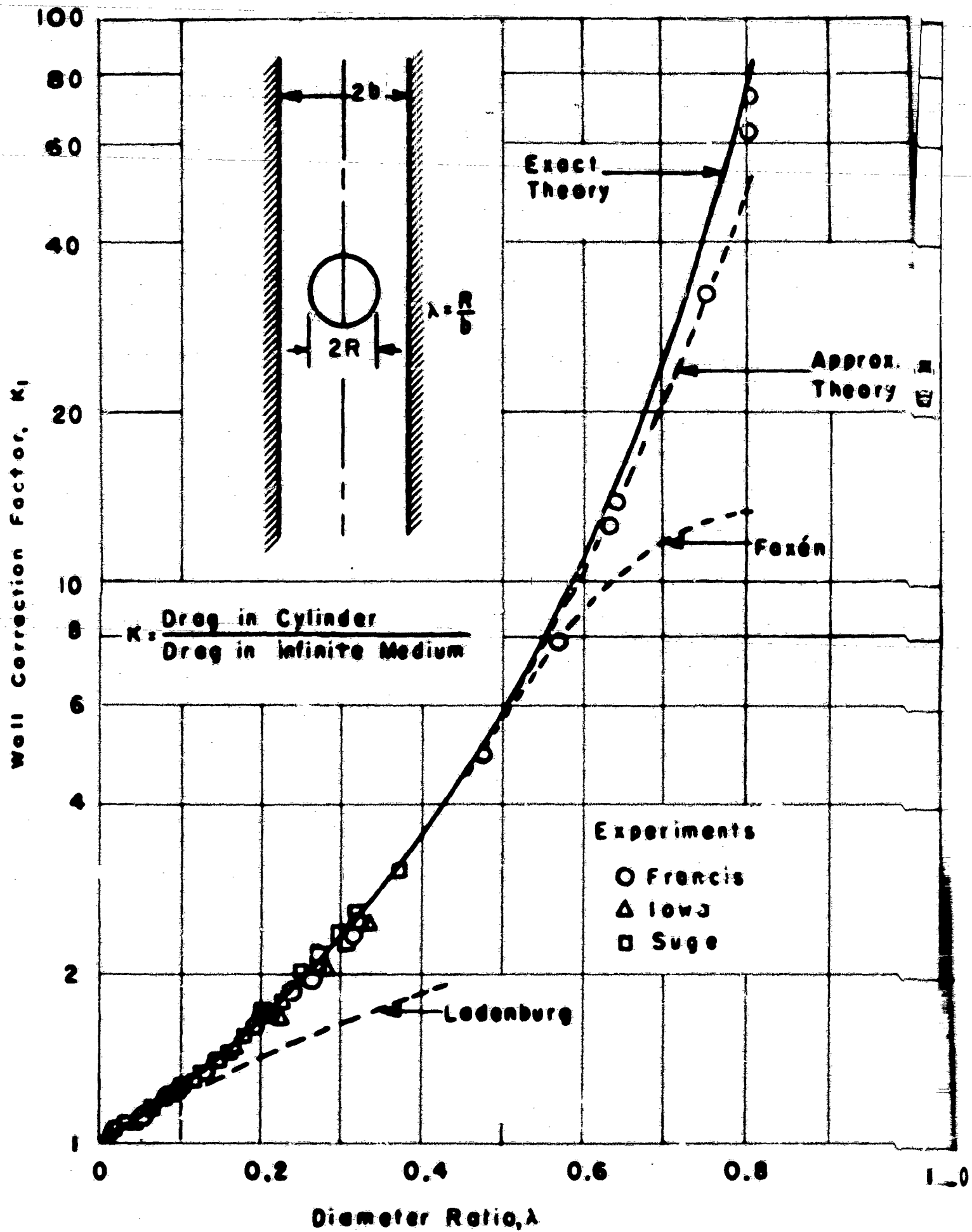


Figure 5 — Wall Correction Factors (K_1) for Rigid Spheres Moving in a Still Liquid Inside a Cylindrical Tube

TABLE 1

Wall Correction Factors (K_1) for Rigid Spheres Moving in a Still Liquid
Inside a Cylindrical Tube

λ	Exact Theory	Approximate Theory (Equation [62])	Percent Difference
0.0	1.000	1.000	0.00
0.1	1.263	1.263	0.00
0.2	1.680	1.680	0.00
0.3	2.371	2.370	- 0.04
0.4	3.596	3.582	- 0.39
0.5	5.970	5.871	- 1.66
0.6	11.135	10.591	- 4.89
0.7	24.955	21.406	-14.22
0.8	73.555	48.985	-33.40

Table 1. Previous theoretical results by Ladenburg and Faxén are included in Figure 5. Experimental data at very low Reynolds numbers, taken from References 9, 22, 23, and 24, are also shown for comparison. Only experimental data at Reynolds number (based on diameter) less than two were included. The actual extent of the Reynolds number range over which the theoretical solutions for the drag of the spheres are valid could not be determined from available experimental data. It is estimated, however, that the theoretical solutions are reliable for Reynolds numbers up to two.

b. Approximate Solution. A very good approximation for the drag of the rigid spheres can be obtained by retaining only the first two equations of the infinite set (Equation [56]) and considering only the constants b_0 and a_1 . For the constant b_0 we then obtain

$$\frac{b_0}{R} = -\frac{3}{2\pi} U \frac{1 - 0.75857 \lambda^5}{1 - 2.1050 \lambda + 2.0865 \lambda^3 - 1.7068 \lambda^5 + 0.72603 \lambda^6} \quad [61]$$

Using the definition of K_1 (Equation [60]), the wall correction factor for rigid spheres in a still liquid within an infinitely long cylinder is then given in the convenient form:

$$K_1 = \frac{1 - 0.75857 \lambda^5}{1 - 2.1050 \lambda + 2.0865 \lambda^3 - 1.7068 \lambda^5 + 0.72603 \lambda^6} \quad [62]$$

*Ladenburg's solution¹: $K = 1 + 2.4 \lambda$

Faxén's solution²: $K = \frac{1}{1 - 2.104 \lambda + 2.09 \lambda^3 - 0.95 \lambda^5}$

Wall correction factors given by Equation [62] have been evaluated and are included in Figure 5. In Table 1, wall correction factors for the exact and approximate solution are listed for comparison. It is seen that the wall correction factors for rigid spheres is well approximated by Equation [62] up to diameter ratios of 0.6.

In this approximation the stream function becomes

$$\Psi(r, \theta) = C_2^{-1/2} (\cos \theta) (A_2 r^2 + B_2 \frac{1}{r} + C_2 r^4 + D_2 r) + C_4^{-1/2} (\cos \theta) (A_4 r^4 + C_4 r^6) + \dots \quad [63]$$

where

$$B_2 = a_1 \pi \quad B_4, B_6, \dots = 0$$

$$D_2 = b_0 \pi \quad D_4, D_6, \dots = 0$$

$$A_2 = -4.40866 \frac{b_0}{R} \lambda - 6.55507 \frac{a_1}{R^3} \lambda^3 + U$$

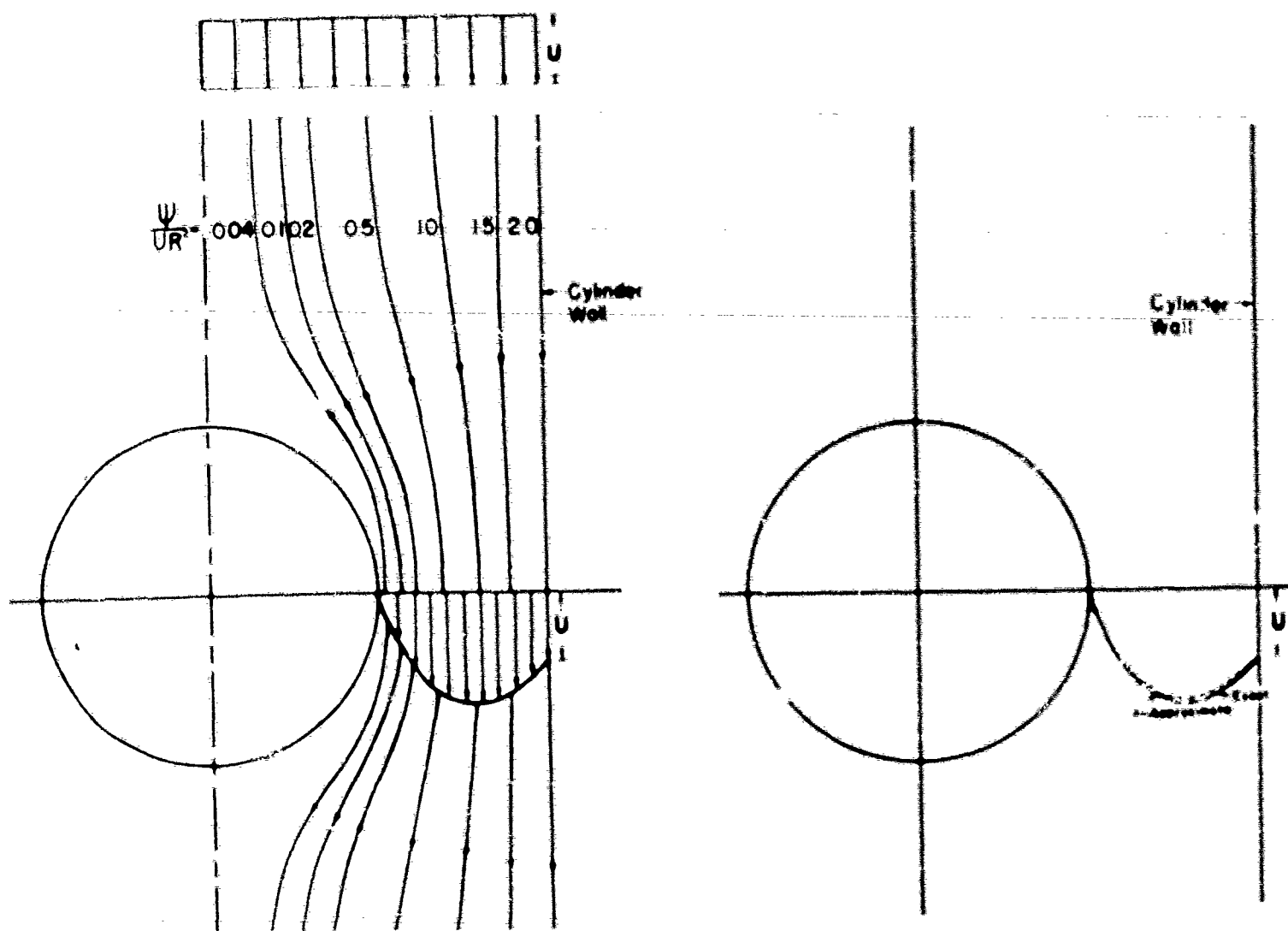
$$A_n = (-1)^{\frac{n}{2}+1} \frac{1}{R^{n-2}} \left\{ \left[\frac{1}{(n-2)!} S_3^{n-1} + \frac{n(n-1)}{(n-2)!(2n-3)} S_4^{n-2} \right] \frac{b_0}{R} \lambda^{n-1} + \left[-\frac{1}{(n-2)!} S_4^n + \frac{1}{(n-4)!(2n-3)} S_2^{n-1} \right] \frac{a_1}{R^3} \lambda^{n+1} \right\}$$

$$C_2 = 1.31101 \frac{b_0}{R^3} \lambda + 3.57466 \frac{a_1}{R^5} \lambda^3$$

$$C_n = (-1)^{\frac{n}{2}+1} \frac{1}{R^n} \frac{1}{(n-2)!(2n+1)} \left(\frac{a_1}{R^3} S_2^{n+1} \lambda^{n+3} + \frac{b_0}{R} S_4^n \lambda^{n+1} \right)$$

In the solution given in Equation [63], all boundary conditions on the cylinder are satisfied. The velocity components arising from the $C_2^{-1/2}$ -terms satisfy the boundary conditions on the sphere. Those from the higher terms ($C_4^{-1/2}, \dots$, etc.) do not satisfy the boundary conditions on the sphere.

Streamlines were evaluated from Equation [63] for a diameter ratio of 0.5, and are shown in Figure 6 for relative motion about the sphere; i.e., as they appear to an observer moving with the sphere.



Sphere in Circular Cylinder, Diameter Ratio: 0.5

Comparison of Velocity Distribution

Figure 5 - Streamlines and Velocity Distribution for Rigid Spheres (Approximate Solution); Relative Motion

2. Fluid Spheres

In this section the axial motion of a fluid body inside an infinitely long cylinder will be considered. For the motion in an infinite medium, it has been shown that the sphere represents a possible surface at which all boundary conditions are satisfied.^{12,13} It will be shown here that the assumption of a spherical shape does not lead to an exact solution for the motion of a fluid body inside a cylinder. However, all velocity and stress components arising from the first group of terms of the stream function (the $C_2^{-1/2}$ terms) do satisfy the boundary conditions on a sphere. Hence, for the case when the $C_2^{-1/2}$ terms are most important in describing the motion, the spherical shape will be a very good approximation. An approximate solution similar to the one for the rigid case will thus be obtained. The validity of this solution is confirmed by experiments.

a. **Theoretical Solution.** As in the rigid case, the origin of the coordinate system is taken at the center of the fluid body. The boundary conditions are:

at the cylinder walls ($\rho = b$): $v_x = -U$ $v_\rho = 0$
 at infinity ($x = \pm \infty$): $v_x = -U$ $v_\rho = 0$
 at the surface of the fluid body: [64]

Equality of tangential velocities,

Normal velocities vanish (i.e., no diffusion),

Equality of tangential and normal forces.

For a spherical surface ($r = R$), the boundary conditions of Equation (6) take the form:

- 1) $u_\theta = v_\theta$
- 2) $u_r = 0$
- 3) $v_r = 0$ [65]
- 4) $(p, \theta)_i = p_{r, \theta}$
- 5) $(p_{rr})_i + \gamma_i g R \cos \theta = p_{rr} + \gamma g R \cos \theta$

where u_r is the radial velocity inside sphere,

u_θ is the tangential velocity inside sphere, and

i refers to quantities inside sphere.

The stream function in the exterior of the sphere is given by [34] i.e.,

$$\Psi(r, \theta) = \sum_2^{\infty} C_n^{-1/2} (\cos \theta) \left(A_n r^n + B_n \frac{1}{r^{n-1}} + C_n r^{n+2} + \dots \right) \quad (n = 2, 4, 6, \dots) \quad [39]$$

in the interior of the sphere by

$$\Psi_i(r, \theta) = \sum_2^{\infty} C_n^{-1/2} (\cos \theta) (E_n r^n + F_n r^{n+2}) \quad (n = 2, 4, 6, \dots) \quad [46]$$

From the above expressions for the stream function and the corresponding velocity components (Equations [18]) and stress components (Equation [17] and [18]), we obtain:

*The $\gamma g R \cos \theta$ and $\gamma_i g R \cos \theta$ terms take the difference in pressure (Figure 7.)

**The constants associated with the $1/r^{n-1}$ and $1/r^{n-3}$ terms are obtained at the center of the sphere.

At static equilibrium, the pressure is constant. (See

the wave function for the velocity would be

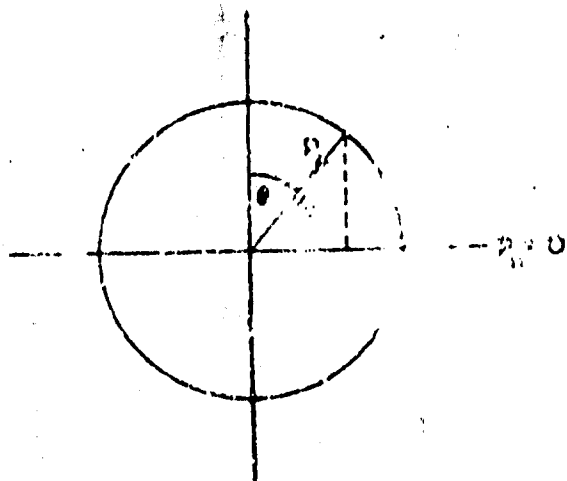


Figure 1 - Hydrostatic Equilibrium in a Fluid Sphere

$$P_{n1} = -\frac{1}{2} \cos \theta$$

$$P_n = -\frac{1}{2} \cos \theta$$

$$u_r = -\sum_2^{\infty} P_{n-1} (E_n r^{n-2} - r^n)$$

$$u_\theta = \sum_2^{\infty} \frac{C_n^{-1/2}}{\sin \theta} [n E_n r^{n-2} - (n+2) F_n r^n]$$

$$u_r = -\sum_2^{\infty} P_{n-1} \left(A_n r^{n-2} - B_n r^{n-1} - C_n r^n + D_n \frac{1}{r^{n-1}} \right)$$

$$u_\theta = \sum_2^{\infty} \frac{C_n^{-1/2}}{\sin \theta} \left[n A_n r^{n-2} - (n-1) B_n \frac{1}{r^{n+1}} + (n+2) C_n r^n - (n-3) D_n \frac{1}{r^{n-1}} \right] \quad (27)$$

$$u_{r\theta} = \mu \sum_2^{\infty} \frac{C_n^{-1/2}}{\sin \theta} \left[2n(n-2) A_n r^{n-3} + 2(n^2-1) B_n \frac{1}{r^{n+2}} + 2(n^2-1) C_n r^{n-1} + 2n(n-2) D_n \frac{1}{r^n} \right]$$

$$p_{rr} = \mu \sum_2^{\infty} P_{n-1} \left\{ -2(n-2) A_n r^{n-3} + 2(n-1) B_n \frac{1}{r^{n+2}} + \left[\frac{2(2n+1)}{n-1} - 2 \right] C_n r^{n-1} + \left[\frac{2(n-3)}{n-1} + 2(n-2) \right] D_n \frac{1}{r^n} \right\}$$

$$(p_{r\theta})_1 = \mu_1 \sum_2^{\infty} \frac{C_n^{-1/2}}{\sin \theta} [2n(n-2) E_n r^{n-3} - 2(n^2-1) F_n r^{n-1}]$$

$$(p_{rr})_1 = \mu_1 \sum_2^{\infty} P_{n-1} \left\{ -2(n-2) E_n r^{n-3} + \left[\frac{2(2n+1)}{n-1} - 2 \right] F_n r^{n-1} \right\} \quad (n=2, 4, 6, \dots)$$

The effect of surface roughness on the average velocity and the time expressions of Equations (1) and (2) is investigated by means of the Legendre polynomial expansion.

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \quad (3)$$

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x), \quad P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \quad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 14x), \quad P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5), \quad P_7(x) = \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 63x), \quad P_8(x) = \frac{1}{128}(6435x^8 - 14620x^6 + 9009x^4 - 1732x^2 + 63), \quad P_9(x) = \frac{1}{128}(12870x^9 - 35277x^7 + 31528x^5 - 10989x^3 + 945x), \quad P_{10}(x) = \frac{1}{256}(170175x^{10} - 554400x^8 + 510315x^6 - 214480x^4 + 35980x^2 - 63).$$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \quad (4)$$

where $P_n(x)$ is the n th Legendre polynomial. The two additional boundary conditions are $u = 0$ at $x = 1$ and $u = 0$ at $x = -1$. For the case of a smooth surface, $u = 0$ at $x = 1$ and $x = -1$ are satisfied. However, if there exist surface roughness, the boundary conditions are not satisfied. In this case, the boundary conditions are modified as follows:

where u_n is the n th order Fourier series coefficient. The boundary conditions are modified as follows:

For the case of a smooth surface, $u = 0$ at $x = 1$ and $x = -1$ are satisfied. However, if there exist surface roughness, the boundary conditions are not satisfied. In this case, the boundary conditions are modified as follows:

$$\begin{aligned} \Psi(x, \rho) = & \int_0^\infty \rho K_1(a\rho) a_1 a \cos ax da + \int_0^\infty \rho^2 K_0(a\rho) b_0 \cos ax da \\ & + \int_0^\infty \left\{ \rho I_1(a\rho) \left[\frac{2a}{(ab)^2} S_3 - S_4 \right] a_1 a + S_3 b_0 \right\} + \rho^2 I_0(a\rho) (S_2 a_1 a + S_4 b_0) \cos ax da \\ & + \frac{U \rho^2}{2} \end{aligned} \quad [69]$$

In spherical coordinates it is given by

$$\Psi(r, \theta) = C_2^{-1/2} (\cos \theta) \left[A_2 r^2 + B_2 \frac{1}{r} + C_2 r^4 + D_2 r \right] + C_4^{-1/2} (\cos \theta) \left[A_4 r^4 + C_4 r^6 \right] + \dots [63]$$

From the boundary conditions on the sphere (Equation [68], $n = 2$)

$$\begin{aligned} A_2 &= -\frac{5}{2} B_2 \frac{1}{R^3} \frac{1}{1-\sigma} - \frac{3}{2} D_2 \frac{1}{R} \frac{1-2/3\sigma}{1-\sigma} \\ C_2 &= \frac{3}{2} B_2 \frac{1}{R^5} \frac{1+2/3\sigma}{1-\sigma} + \frac{1}{2} D_2 \frac{1}{R^3} \frac{1}{1-\sigma} \\ E_2 &= -\frac{5}{2} B_2 \frac{1}{R^3} \frac{\sigma}{1-\sigma} - \frac{1}{2} D_2 \frac{1}{R} \frac{\sigma}{1-\sigma} \\ F_2 &= -\frac{E_2}{R^2} \end{aligned} \quad [70]^*$$

where $\sigma = \frac{\text{external viscosity}}{\text{internal viscosity}}$.

Substituting expressions for A_2 , C_2 , B_2 , and D_2 (Equation [68]) into Equation [70], two algebraic equations for a_1 and b_0 are obtained

*A special set of equations is obtained for the case $\sigma = 1$, namely:

$$\begin{aligned} A_2 + C_2 R^2 + \frac{4}{3} D_2 R^{-1} &= 0 \\ B_2 &= -\frac{1}{3} D_2 R^2 \\ E_2 &= A_2 + D_2 R^{-1} \\ F_2 &= -E_2 R^{-2} \end{aligned}$$

Use of these equations leads to the same result for K_1 as putting $\sigma = 1$ into Equation [75].

$$\frac{b_0}{R} \left(\frac{3}{2} \pi \frac{1 - 2/3 \sigma}{1 - \sigma} - 4.40866 \lambda \right) + \frac{a_1}{R^3} \left(\frac{5}{2} \pi \frac{1}{1 - \sigma} - 6.55507 \lambda^3 \right) = -U \quad [71]$$

$$\frac{b_0}{R} \left(\frac{5}{2} \pi \frac{1}{1 - \sigma} - 6.55507 \lambda^3 \right) + \frac{a_1}{R^3} \left(\frac{15}{2} \pi \frac{1 + 2/3 \sigma}{1 - \sigma} - 17.8733 \lambda^5 \right) = 0$$

From the above equations b_0 , for example, becomes

$$\frac{b_0}{R} = - \frac{3}{2\pi} \frac{1 + 2/3 \sigma}{1 + \sigma} U \quad [72]$$

$$\frac{1 - 0.75857 \frac{1 - \sigma}{1 + 2/3 \sigma} \lambda^5}{1 - 2.1050 \frac{1 + 2/3 \sigma}{1 + \sigma} \lambda + 2.0865 \frac{1}{1 + \sigma} \lambda^3 - 1.7068 \frac{1 - 2/3 \sigma}{1 + \sigma} \lambda^5 + 0.72603 \frac{1 - \sigma}{1 + \sigma} \lambda^6}$$

The constants b_0 and a_1 , as determined from Equation [71], when substituted in Equation [63] will yield the exterior stream function for the fluid sphere. In the interior of the sphere the stream function can be given as (utilizing velocity continuity):

$$\Psi_i(r, \theta) = C_2^{-1/2} (\cos \theta) E_2 r^2 \left(1 - \frac{r^2}{R^2} \right) + C_4^{-1/2} (\cos \theta) (A_4 r^4 + C_4 r^6) + \dots \quad [73]$$

Streamlines for the interior and exterior of the fluid sphere were evaluated from Equations [73] and [63] (with coefficients b_0 and a_1 determined from Equation [71]) for the case of an infinite viscosity ratio and a diameter ratio of 0.5. They are shown in Figures 8 and 9. Figure 8 shows the relative motion; Figure 9 gives the absolute motion. Velocity distributions at the plane of symmetry and streamlines for the motion of a corresponding sphere in an infinite medium have been included in both figures.*

Using Equations [58] and [72], we obtain

$$\boxed{\text{Drag} = 6 \pi \mu R \frac{1 + 2/3 \sigma}{1 + \sigma} K_1} \quad [74]$$

Drag in Infinite Medium

*The stream function for the motion of fluid spheres in an infinite medium is:

$$\Psi(r, \theta) = \frac{1}{2} \sin^2 \theta \left[U r^2 + \frac{1}{2} U R^3 \frac{1}{1 + \sigma} \frac{1}{r} - \frac{3}{2} U R \frac{1 + 2/3 \sigma}{1 + \sigma} r \right] \quad \Psi_i(r, \theta) = \frac{1}{2} \sin^2 \theta \left[\frac{U}{2} \frac{\sigma}{1 + \sigma} r^2 \left(\frac{r^2}{R^2} - 1 \right) \right] \quad \text{Relative motion}$$

$$\Psi(r, \theta) = \frac{1}{2} \sin^2 \theta \left[\frac{1}{2} U R^3 \frac{1}{1 + \sigma} \frac{1}{r} - \frac{3}{2} U R \frac{1 + 2/3 \sigma}{1 + \sigma} r \right] \quad \Psi_i(r, \theta) = \frac{1}{2} \sin^2 \theta \left[-U r^2 + \frac{U}{2} \frac{\sigma}{1 + \sigma} r^2 \left(\frac{r^2}{R^2} - 1 \right) \right] \quad \text{Absolute motion}$$

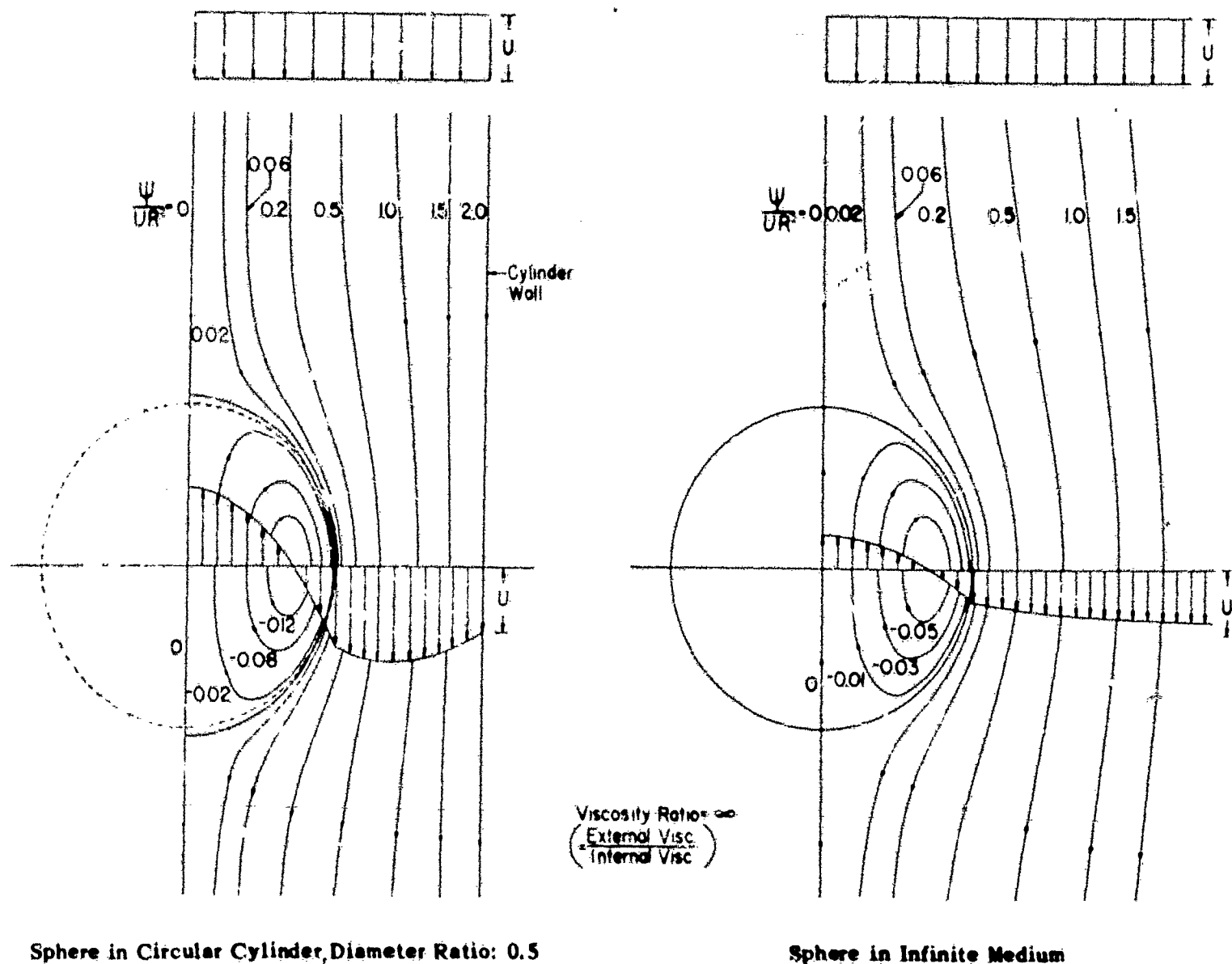


Figure 8 – Streamlines and Velocity Distribution for Fluid Spheres; Relative Motion

where K_1 , the wall correction factor for fluid spheres moving in a still liquid inside an infinitely long cylinder, is

$$K_1 = \frac{1 - 0.75857 \frac{1 - \sigma}{1 + 2/3 \sigma} \lambda^5}{1 - 2.1050 \frac{1 + 2/3 \sigma}{1 + \sigma} \lambda + 2.0865 \frac{1}{1 + \sigma} \lambda^3 - 1.7068 \frac{1 - 2/3 \sigma}{1 + \sigma} \lambda^5 + 0.72603 \frac{1 - \sigma}{1 + \sigma} \lambda^6} \quad [75]^*$$

Wall correction factors have been evaluated from Equation [75] and are given in Figure 10. Curves for fluid spheres of viscosity ratio 0 (rigid case), 1, 13, and ∞ are shown.

*For $\sigma = 0$, the expression for K_1 reduces to that of Equation [62].

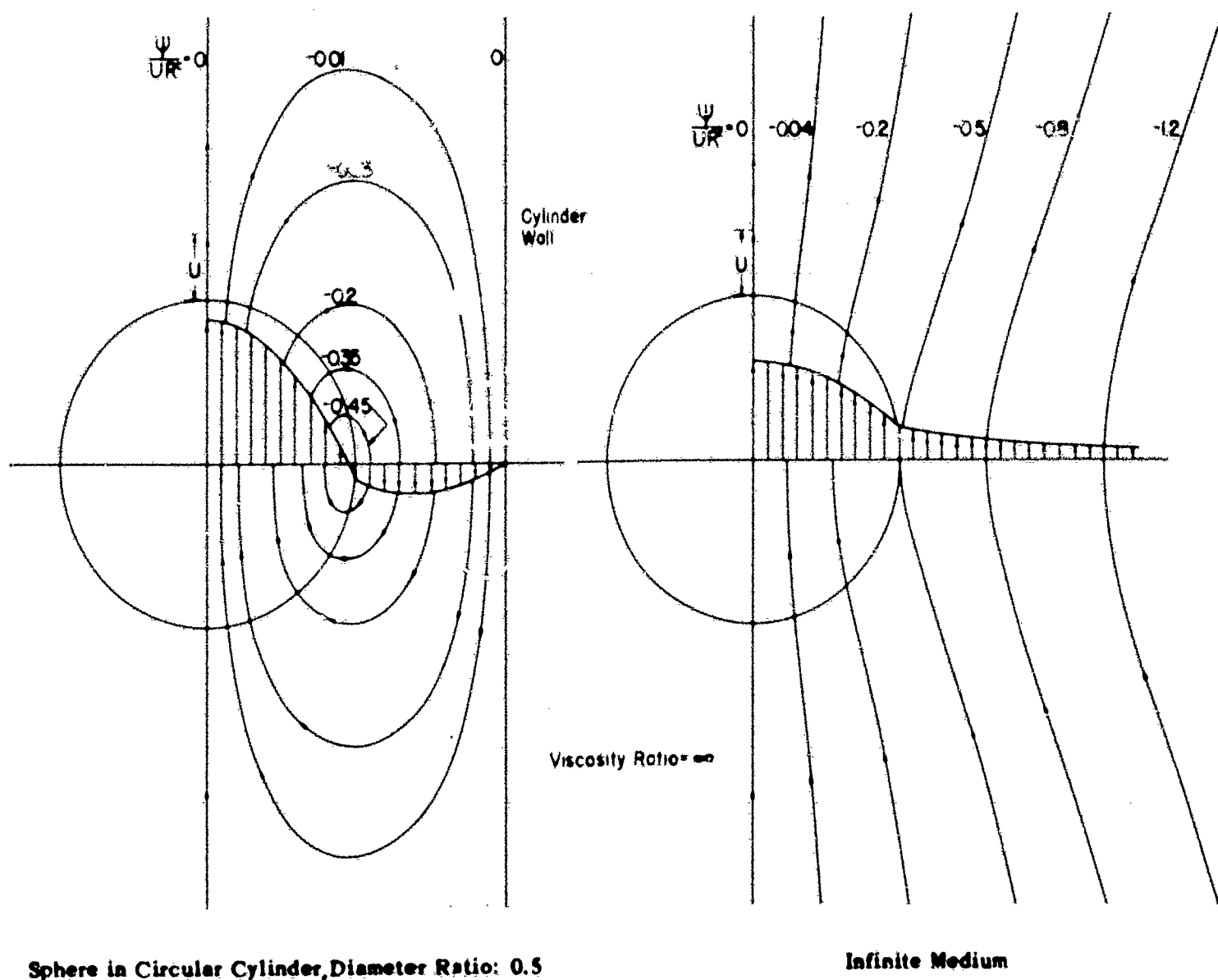


Figure 9 – Streamlines and Velocity Distribution for Fluid Spheres; Absolute Motion

b. Experimental Work. To check the validity of the theoretical solution for the drag of fluid spheres and to determine the deviation of fluid bodies in a cylindrical container from the spherical shape, a limited number of experiments was conducted. The experimental study consisted of determining the rate of fall and the size of water-glycerine and Dow-Corning 200 silicone drops in castor oil. The DC 200 corresponded to viscosity ratio of 13, whereas the water-glycerine mixture (about 45 percent glycerine) gave a ratio of about 200. Two cylindrical Lucite tubes were used. They were 1.24 in. and 2.74 in. in internal diameter and 36 in. and 31 in. in length, respectively. The drops were generated by means of a stopcock burette, the tip of which protruded into the castor oil.* By regulating the stopcock, drops of different size were released. Large drops were used to assure that they behaved as "fluid bodies." The fall velocities of the drops were determined by means of a stop watch; the size and shape

*The use of a squeeze bulb filled with oil proved unsatisfactory due to the injection of small air bubbles into the drop. The small bubbles, moving within the drop however, showed clearly the circulatory motion within the drop.

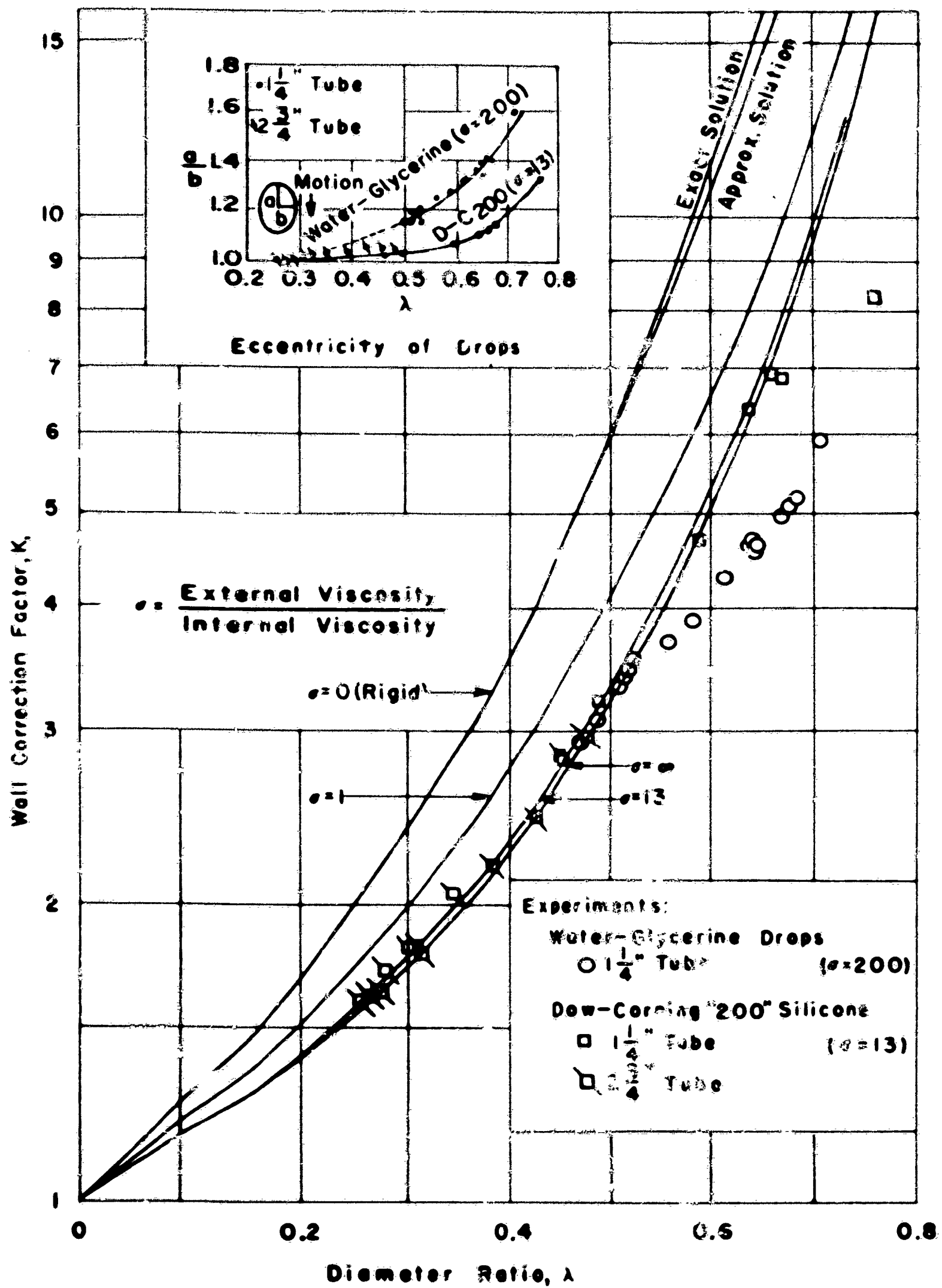


Figure 10 — Wall Correction Factors for Spheres Moving in a Still Liquid Inside a Cylindrical Tube

of the drops were determined from high-speed photographs. Stationary steel spheres of various known diameters, located at the center of the tubes, were photographed to provide scale factors for evaluating size and shape. The temperature of the castor oil was determined, at frequent intervals, by means of an immersion thermometer. The viscosity of the liquids was measured with an Ostwald-Fenske viscometer, and the density of the liquids was obtained by means of a Westphal specific-gravity balance.

The experimental drag of the drops was determined from the equilibrium condition of the forces acting on them. Thus

$$\text{Drag} = \underbrace{\text{Volume } \gamma_i g}_{\text{Weight}} - \underbrace{\text{Volume } \gamma g}_{\text{Buoyancy}}$$

The experimentally determined drag can be compared with the theoretical one given in Equation [74]. To obtain the experimental wall correction factors, the experimental drags in the cylindrical tubes were divided by the drag of the fluid spheres in an infinite medium. For deformed drops, an equivalent radius $\left[= \left(\frac{\text{volume}}{4/3 \pi} \right)^{1/3} \right]$ was used in the computations. The experimental correction factors thus determined are shown in Figure 10. The drops as determined from the experiments were either spheres or ellipsoids of revolution with major axes in the direction of motion. The eccentricity of the drops is given in the insert of Figure 10. The maximum Reynolds number (based on diameter) of the water-glycerine drops was 0.72; the maximum for the DC 200 drops was 0.2. It is seen from this figure that the theoretical solution agrees well with experimental results for diameter ratios up to about 0.5. The deviation from the theoretical curves occurs at diameter ratio of 0.53 for the water-glycerine drops and at a ratio of 0.65 for the DC 200 drops. Because of the limited range of the experiments, no definite conclusion can be drawn regarding this difference. It is conceivable, however, that the deviation from the approximate theory is a function of the viscosity ratio σ ; i.e., the smaller this ratio the larger the diameter ratio at which the theory deviates from the experimental results. Although the largest σ used was 200, it is not believed that the deformation of bodies with $\sigma \rightarrow \infty$ would be much more severe. It is further noticed from Figure 10 that the wall correction factor is not too sensitive to deformation from the spherical shape. A difference in the axes of the drops of up to about 15 percent occurred without noticeably affecting the wall correction factor.

B. MOTION IN A MOVING LIQUID

The preceding sections have dealt with the steady, axial translation of rigid and fluid spheres in a stationary, viscous, incompressible fluid bounded by an infinitely long cylinder. The solutions are now extended to include the case in which the fluid contained within the cylinder is also in motion; i.e., a parabolic velocity distribution (Poiseuille flow) exists at infinity. As before, two problems are considered; the first deals with the motion of rigid spheres, the second with fluid spheres. The method of solution is the same as before; the only difference arises from the changed boundary conditions at infinity.

An exact and approximate solution is given for the drag of moving rigid spheres in a moving liquid. The approximate solution is shown to be valid for diameter ratios up to about 0.6. An approximate solution is given for fluid spheres and is estimated to be valid for diameter ratios up to about 0.5. It is shown that the drag of a sphere in motion within a moving liquid is composed of two parts: namely, the drag due to the motion of the sphere in a still liquid, and the drag due to the motion of the liquid within the cylindrical tube past a stationary sphere. Values of wall correction factors have thus been computed for the two special cases. The drag of moving spheres within a moving liquid can be obtained by appropriate combination of these two correction factors.

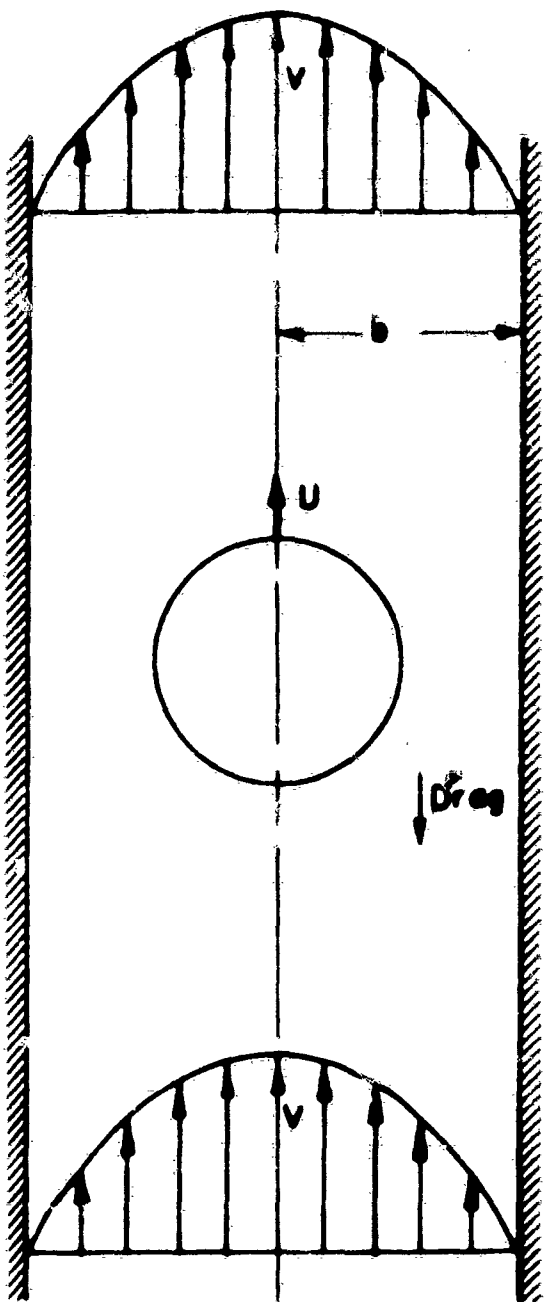
1. Rigid Spheres

a. **Exact Solution.** The problem considered here is that of a rigid sphere moving within a fixed circular cylinder containing a fluid having a parabolic velocity distribution at infinity. For convenience, it is desired to take the coordinate system fixed with respect to the sphere. Thus an equivalent system, as indicated in Figure 11, is used. The coordinate origin is again taken at the center of the sphere, and the cylinder is assumed to be moving at constant velocity U in the negative x -direction. The boundary conditions are:

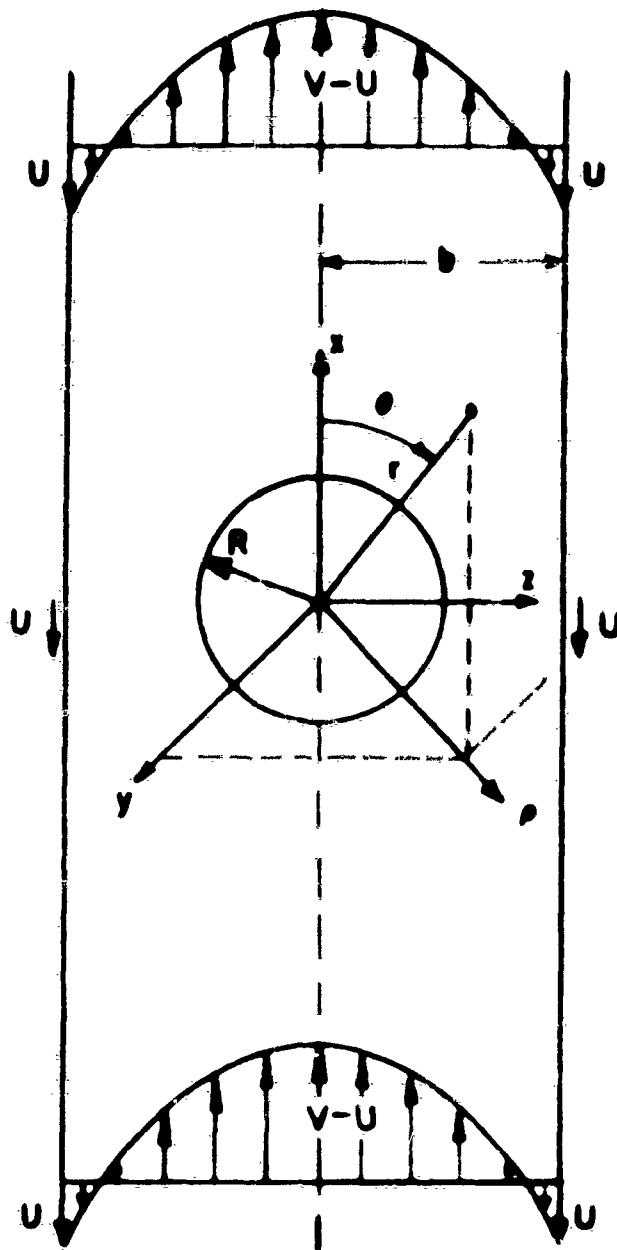
$$\begin{aligned} \text{at the cylinder walls } (\rho = b): \quad & v_x = -U; \quad v_\rho = 0 \quad \text{or} \quad \Psi = \frac{U}{2} b^2 - \frac{V}{4} b^2 \\ \text{at infinity } (x = \pm\infty): \quad & v_x = -U + V \left(1 - \frac{\rho^2}{b^2}\right); \quad v_\rho = 0 \\ \text{on the surface of the sphere } (r = R): \quad & v_r = 0; \quad v_\theta = 0 \end{aligned} \quad [76]$$

After satisfying the boundary conditions on the cylinder walls, the stream function becomes (similar to Equation [45])

$$\begin{aligned} \Psi(x, \rho) = \int_0^\pi \{ \rho K_1(a\rho) f_1(a) + \rho^2 K_0(a\rho) F_1(a) + \rho I_1(a\rho) [S_1 f_1(a) + S_3 F_1(a)] + \\ + \rho^2 I_0(a\rho) [S_2 f_1(a) + S_4 F_1(a)] \} \cos ax da + \left(\frac{U}{2} - \frac{V}{2} \right) \rho^2 + \frac{V}{4b^2} \rho^4 \end{aligned} \quad [77]$$



Sphere Moving with Velocity U in Moving Liquid
Inside Fixed Cylindrical Tube



Equivalent System: Sphere Fixed; Tube Moving with
Velocity U . Velocity $-U+V (1-\rho^2/b^2)$ at Infinity

Figure 11 — Definition Sketch

As before, the stream function Ψ is now rewritten as

$$\begin{aligned}
 \Psi = & a_1 \frac{\pi}{r} C_2^{-1/2} - a_3 \frac{12\pi}{r^3} C_4^{-1/2} + a_5 \frac{360\pi}{r^5} C_6^{-1/2} - \dots + \\
 & + \frac{1}{2} \rho^2 \left[- \int_0^\infty S_4 (a_1 a^2 + a_3 a^4 + \dots) da + \int_0^\infty (S_3 a + 2S_4) (b_0 + b_2 a^2 + \dots) da + (U-V) \right] \\
 & + \frac{1}{4} x^2 \rho^2 \left[\int_0^\infty S_4 (a_1 a^4 + a_3 a^6 + \dots) da - \int_0^\infty (S_3 a + 2S_4) (b_0 a^2 + b_2 a^4 + \dots) da \right] \quad [78] \\
 & + \frac{1}{24} \rho^4 \left[\int_0^\infty \left(3S_2 - \frac{3}{2} S_4 a \right) (a_1 a^3 + a_3 a^5 + \dots) da + \int_0^\infty \left(\frac{3}{2} S_3 a + 6S_4 \right) (b_0 a^2 + \right. \\
 & \left. + b_2 a^4 + \dots) da - 6 \frac{V}{b^2} \right] + \dots
 \end{aligned}$$

Again, comparison with the solution for Ψ in spherical coordinates (Equation [52]) results in the following relationships between the constants A_n , B_n , C_n , D_n , and a_n , b_n :

$$\begin{aligned}
 B_2 &= a_1 \pi + \frac{2}{5} b_2 \pi; & B_4 &= -(12a_3 + 16b_4) \pi; \dots \\
 D_2 &= b_0 \pi; & D_4 &= -\frac{12}{5} b_2 \pi; \dots \\
 A_2 &= -6.55507 \frac{\lambda^3}{R^3} a_1 - 15.3046 \frac{\lambda^5}{R^5} a_3 - \dots - 4.40866 \frac{\lambda}{R} b_0 - 4.76313 \frac{\lambda^3}{R^3} b_2 - \dots + (U - V) \\
 C_2 &= 3.57466 \frac{\lambda^5}{R^5} a_1 + 12.1172 \frac{\lambda^7}{R^7} a_3 + \dots + 1.31101 \frac{\lambda^3}{R^3} b_0 + 3.06092 \frac{\lambda^5}{R^5} b_2 + \dots + \frac{2}{5} \frac{V}{b^2} \\
 A_4 &= 4.07765 \frac{\lambda^5}{R^5} a_1 + 54.7950 \frac{\lambda^7}{R^7} a_3 + \dots + 1.07056 \frac{\lambda^3}{R^3} b_0 + 11.9275 \frac{\lambda^5}{R^5} b_2 + \dots - \frac{2}{5} \frac{V}{b^2}
 \end{aligned} \tag{79}$$

etc.

As previously (Equation [55])

$$\begin{aligned}
 A_n &= -\frac{2n+1}{2} B_n \frac{1}{R^{2n-1}} - \frac{2n-1}{2} D_n \frac{1}{R^{2n-3}} \\
 C_n &= \frac{2n-1}{2} B_n \frac{1}{R^{2n+1}} + \frac{2n-3}{2} D_n \frac{1}{R^{2n-1}}
 \end{aligned} \tag{55}$$

Substituting the expressions for A_n , B_n , C_n , D_n as given in Equation [79] into [55], the infinite set for the determination of the constants a_1 , a_3 , ... and b_0 , b_2 , ... is obtained

from A_2 :

$$\frac{b_0}{R} \left(\frac{3}{2} \pi - 4.40866 \lambda \right) + \frac{b_2}{R^3} (\pi - 4.76313 \lambda^3) + \dots + \frac{a_1}{R^3} \left(\frac{5}{2} \pi - 6.55507 \lambda^3 \right) + \frac{a_3}{R^5} (-15.3046 \lambda^5) + \dots = -U + V$$

from C_2 :

$$\frac{b_0}{R} \left(\frac{3}{2} \pi - 6.55507 \lambda^3 \right) + \frac{b_2}{R^3} (3\pi - 15.3046 \lambda^5) + \dots + \frac{a_1}{R^3} \left(\frac{15}{2} \pi - 17.8733 \lambda^5 \right) + \frac{a_3}{R^5} (-60.5861 \lambda^7) + \dots = 2V \lambda^2$$

from A_4 :

$$\frac{b_0}{R} (-1.07056 \lambda^3) + \frac{b_2}{R^3} \left(\frac{42}{5} \pi - 11.9275 \lambda^5 \right) + \dots + \frac{a_1}{R^3} (-4.07765 \lambda^5) + \frac{a_3}{R^5} (54\pi - 54.7950 \lambda^7) + \dots = -\frac{2}{5} V \lambda^2$$

from C_4 :

$$\frac{b_0}{R} (-0.85026 \lambda^5) + \frac{b_2}{R^3} (6\pi - 7.43469 \lambda^7) + \dots + \frac{a_1}{R^3} (-3.36589 \lambda^7) + \frac{a_3}{R^5} (42\pi - 37.5381 \lambda^9) + \dots = 0 \quad [80]$$

etc.

As in the case of the motion of a sphere in a still liquid, a wall correction factor could be defined based on the drag in an infinite medium [i.e., $6\pi\mu R(U - V)$]. However, the wall correction factor thus obtained would be a function of the diameter ratio λ and the velocity ratio U/V . It is more convenient to define a correction factor which is a function of λ only.

The coefficient b_0 as obtained from Equations [80] can be written (utilizing properties of determinants) as:

$$b_0 = b_{01} - b_{02}$$

where b_{01} is determined from System 1 and b_{02} from System 2

System 1	System 2
$\frac{b_0}{R} \left(\frac{3}{2} \pi - 4.40866 \lambda \right) + \dots = -U$	$\frac{b_0}{R} \left(\frac{3}{2} \pi - 4.40866 \lambda \right) + \dots = -V$
$\frac{b_0}{R} \left(\frac{5}{2} \pi - 6.55507 \lambda^3 \right) + \dots = 0$	$\frac{b_0}{R} \left(\frac{5}{2} \pi - 6.55507 \lambda^3 \right) + \dots = -2V \lambda^2$
$\frac{b_0}{R} (-1.07056 \lambda^3) + \dots = 0$	$\frac{b_0}{R} (-1.07056 \lambda^3) + \dots = \frac{2}{5} V \lambda^2$
etc.	etc.

As given previously, the drag experienced by a sphere is

$$\text{Drag} = -\frac{4}{3} \pi \mu b_0 \quad [58]$$

where the drag is taken positive in the negative x -direction as indicated in Figure 11. We now define two wall correction factors K_1 and K_2 such that

$$K_1 = -\frac{4\pi^2 \mu b_{01}}{6\pi \mu UR} = -\frac{2\pi}{3} \frac{b_{01}}{UR} \quad [60]$$

$$K_2 = -\frac{4\pi^2 \mu b_{02}}{6\pi \mu VR} = -\frac{2\pi}{3} \frac{b_{02}}{VR} \quad [82]$$

TABLE 2

Wall Correction Factors (K_2) for Fixed Rigid Spheres in Poiseuille Flow

λ	Exact Theory	Approximate Theory (Equation [16])	Percent Difference
0.0	1.000	1.000	0.00
0.1	1.255	1.255	0.00
0.2	1.635	1.635	0.00
0.3	2.231	2.231	0.00
0.4	3.218	3.218	0.00
0.5	5.004	4.973	- 0.83
0.6	8.651	8.377	- 3.17
0.7	17.671	15.666	-11.25
0.8	47.301	33.056	-30.1

hence

$$\text{Drag} = 6 \pi \mu R (U K_1 - V K_2)$$

[83]

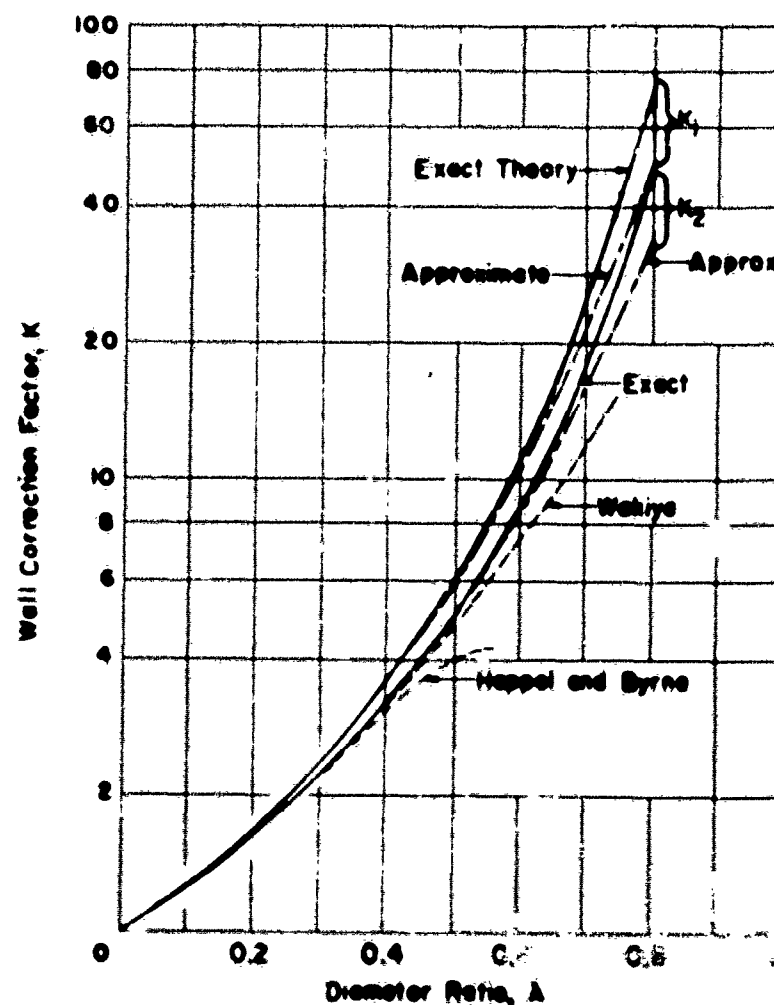


Figure 12 -- Wall Correction Factors for Rigid Spheres within a Cylinder

Thus the drag of a sphere in motion within a moving liquid is composed of two parts: (1) the drag due to the motion of the sphere with velocity U in a still liquid, and (2) the drag due to the sphere held fixed within a moving liquid having a parabolic velocity distribution (maximum velocity V) at infinity.

The coefficients b_{01} have been determined in a previous section from Equations [56]. The coefficients b_{02} were determined numerically from System 2, Equation [81], in the same manner as b_{01} over a range of diameter ratios λ . The computed wall correction factors K_1 (for rigid spheres moving in a still liquid inside an infinitely long cylinder) were shown in Table 1. Table 2 gives the computed wall correction factors K_2 (for fixed rigid spheres within Poiseuille flow). Both wall correction factors are shown in Figure 12.

The stream function in spherical coordinates for flow about a moving rigid sphere

in Poiseuille flow is given by Equation [39] with coefficients A_n, B_n, C_n, D_n as determined from Equation [79] (with coefficients a_n and b_n evaluated from Equations [80] or [81]).

b. Approximate Solution. Again, an approximate solution for the rigid spheres can be obtained by retaining only the coefficients b_0 and a_1 . From Equation [80] we obtain

$$\frac{b_0}{R} = -\frac{3}{2\pi} \frac{(U-V)(1-0.75857\lambda^5) + V\left(\frac{2}{3}\lambda^2 - 0.55640\lambda^5\right)}{1-2.1050\lambda + 2.0865\lambda^3 - 1.7068\lambda^5 + 0.72603\lambda^6} \quad [84]$$

The drag of the moving rigid spheres in a moving liquid becomes

$$\text{Drag} = 6\pi\mu R \frac{(U-V)(1-0.75857\lambda^5) + V\left(\frac{2}{3}\lambda^2 - 0.55640\lambda^5\right)}{1-2.1050\lambda + 2.0865\lambda^3 - 1.7068\lambda^5 + 0.72603\lambda^6} \quad [85]$$

Using definitions of K_1 and K_2 (Equations [60] and [82]), we obtain

$$K_1 = \frac{1-0.75857\lambda^5}{1-2.1050\lambda + 2.0865\lambda^3 - 1.7068\lambda^5 + 0.72603\lambda^6} \quad [62]$$

$$K_2 = \frac{1 - \frac{2}{3}\lambda^2 - 0.20217\lambda^5}{1-2.1050\lambda + 2.0865\lambda^3 - 1.7068\lambda^5 + 0.72603\lambda^6} \quad [86]$$

Wall correction factors as obtained from Equations [62] and [86] are shown in Figure 12 and Tables 1 and 2. Previous theoretical results by Happel and Byrne⁶ and Wakiya⁵ (presented in terms of K_2) are also included in Figure 12.

It is seen from the tables that the approximate expressions for K_1 and K_2 agree well with the exact solution up to diameter ratios of 0.6. However, for certain combinations of $UK_1 - VK_2$ (e.g., $U/V = 3/4$), the range of good agreement will decrease to diameter ratios of about 0.5.

The approximate solution for the stream function in spherical coordinates for flow about a moving rigid sphere in Poiseuille flow is given by Equation [63] with coefficients A_n, B_n, C_n, D_n as follows:

Happel and Byrne's solution:⁶

$$\text{Drag} = 6\pi\mu R \frac{(U-V) + V\frac{2}{3}\lambda^2}{1-2.105\lambda + 2.087\lambda^3}$$

Wakiya's solution:⁵

$$\text{Drag} = 6\pi\mu R \frac{-V + V\frac{2}{3}\lambda^2}{1-2.105\lambda + 2.09\lambda^3 - 1.11\lambda^5}$$

$$\Psi(r, \theta) = C_2^{-1/2} (\cos \theta) \left(A_2 r^2 + B_2 \frac{1}{r} + C_2 r^4 + D_2 r \right) + C_4^{-1/2} (\cos \theta) (A_4 r^4 + C_4 r^6) + \dots$$

$$B_2 = a_1 \pi \quad B_4, B_6, \dots = 0$$

$$D_2 = b_0 \pi \quad D_4, D_6, \dots = 0$$

$$A_2 = -4.40866 \frac{b_0}{R} \lambda - 6.55507 \frac{a_1}{R^3} \lambda^3 + U - V$$

[87]

$$C_2 = 1.31101 \frac{b_0}{R^3} \lambda^3 + 3.57466 \frac{a_1}{R^5} \lambda^5 + \frac{2}{5} \frac{V}{b^2}$$

$$A_4 = 1.07056 \frac{b_0}{R^3} \lambda^3 + 4.07765 \frac{a_1}{R^5} \lambda^5 - \frac{2}{5} \frac{V}{b^2}$$

$$\left. \begin{array}{l} C_n = \\ A_n = \end{array} \right\} \text{ same as in Equation [63]}$$

2. Fluid Spheres

For fluid bodies moving within Poiseuille flow, the same type of approximation is utilized as for the motion of a fluid sphere in a still liquid. As before, the origin of the coordinate system is taken at the center of the fluid body. The boundary conditions are:

$$\text{at the cylinder walls } (\rho = b): \quad v_x = -U; \quad v_\rho = 0$$

$$\text{at infinity } (x = \pm \infty): \quad v_x = -U + V \left(1 - \frac{\rho^2}{b^2} \right); \quad v_\rho = 0$$

$$\text{at the surface of the fluid body:} \quad [88]$$

Equality of tangential velocities

Normal velocities vanish (i.e., no diffusion)

Equality of tangential and normal forces.

Again assuming a resultant spherical shape for the fluid body, we obtain, as before, from the boundary conditions on the sphere

$$\begin{aligned} A_2 &= -\frac{5}{2} B_2 \frac{1}{R^3} \frac{1}{1-\sigma} - \frac{3}{2} D_2 \frac{1}{R} \frac{1 - \frac{2}{3}\sigma}{1-\sigma} & E_2 &= -\frac{5}{2} B_2 \frac{1}{R^3} \frac{\sigma}{1-\sigma} - \frac{1}{2} D_2 \frac{1}{R} \frac{\sigma}{1-\sigma} \\ C_2 &= \frac{3}{2} B_2 \frac{1}{R^5} \frac{1 + \frac{2}{3}\sigma}{1-\sigma} + \frac{1}{2} D_2 \frac{1}{R^3} \frac{1}{1-\sigma} & F_2 &= \frac{E^2}{R^2} \end{aligned} \quad [70]$$

Substituting expressions for A_2 , C_2 , B_2 , and D_2 (as given in Equation [87]) into Equation [7], two algebraic equations for b_0 and a_1 are obtained

$$\frac{b_0}{R} \left(\frac{3}{2} \pi \frac{1 - \frac{2}{3} \sigma}{1 - \sigma} - 4.40866 \lambda \right) + \frac{a_1}{R} \left(\frac{5}{2} \frac{1}{1 - \sigma} - 6.55507 \lambda^3 \right) = -(U - V) \quad [88]$$

$$\frac{b_0}{R} \left(\frac{5}{2} \pi \frac{1}{1 - \sigma} - 6.55507 \lambda^3 \right) + \frac{a_1}{R} \left(\frac{15}{2} \pi \frac{1 + \frac{2}{3} \sigma}{1 - \sigma} - 17.8733 \lambda^3 \right) = 2V \lambda^2$$

From the above equations b_0 and a_1 become

$$\frac{b_0}{R} = \frac{3}{2\pi} \frac{1 + \frac{2}{3} \sigma}{1 + \sigma} \frac{(U - V) \left(1 - 0.75857 \frac{1 - \sigma}{1 + 2/3 \sigma} \lambda^5 \right) + V \left(\frac{2}{3} \frac{1}{1 + 2/3 \sigma} \lambda^2 - 0.55640 \frac{1 - \sigma}{1 + 2/3 \sigma} \lambda^5 \right)}{1 - 2.1050 \frac{1 + 2/3 \sigma}{1 + \sigma} \lambda + 2.0865 \frac{1}{1 + \sigma} \lambda^3 - 1.7068 \frac{1 - 2/3 \sigma}{1 + \sigma} \lambda^5 + 0.72603 \frac{1 - \sigma}{1 + \sigma} \lambda^6} \quad [90]$$

$$\frac{a_1}{R^3} = \frac{1}{2\pi} \frac{1}{1 + \sigma} \frac{(U - V) [1 - 0.83462 (1 - \sigma) \lambda^3] + V \left[\frac{6}{5} \lambda^2 - 1.12266 (1 - \sigma) \lambda^3 \right]}{1 - 2.1050 \frac{1 + 2/3 \sigma}{1 + \sigma} \lambda + 2.0865 \frac{1}{1 + \sigma} \lambda^3 - 1.7068 \frac{1 - 2/3 \sigma}{1 + \sigma} \lambda^5 + 0.72603 \frac{1 - \sigma}{1 + \sigma} \lambda^6}$$

The drag of the moving fluid spheres in a moving liquid then becomes

$$\text{Drag} = 6\pi\mu R \frac{1 + \frac{2}{3} \sigma}{1 + \sigma} \frac{(U - V) \left(1 - 0.75857 \frac{1 - \sigma}{1 + 2/3 \sigma} \lambda^5 \right) + V \left(\frac{2}{3} \frac{1}{1 + 2/3 \sigma} \lambda^2 - 0.55640 \frac{1 - \sigma}{1 + 2/3 \sigma} \lambda^5 \right)}{1 - 2.1050 \frac{1 + 2/3 \sigma}{1 + \sigma} \lambda + 2.0865 \frac{1}{1 + \sigma} \lambda^3 - 1.7068 \frac{1 - 2/3 \sigma}{1 + \sigma} \lambda^5 + 0.72603 \frac{1 - \sigma}{1 + \sigma} \lambda^6} \quad [91]$$

Defining

$$K_1 = - \frac{4\pi^2 \mu b_{01}}{6\pi\mu UR \frac{1 + 2/3 \sigma}{1 + \sigma}} = - \frac{2\pi}{3} \frac{1 + \sigma}{1 + \frac{2}{3} \sigma} \frac{b_{01}}{UR} \quad [92]$$

and

$$K_2 = - \frac{4\pi^2 \mu b_{02}}{6\pi\mu VR \frac{1 + 2/3 \sigma}{1 + \sigma}} = - \frac{2\pi}{3} \frac{1 + \sigma}{1 + \frac{2}{3} \sigma} \frac{b_{02}}{VR} \quad [93]$$

the wall correction factors K_1 and K_2 become

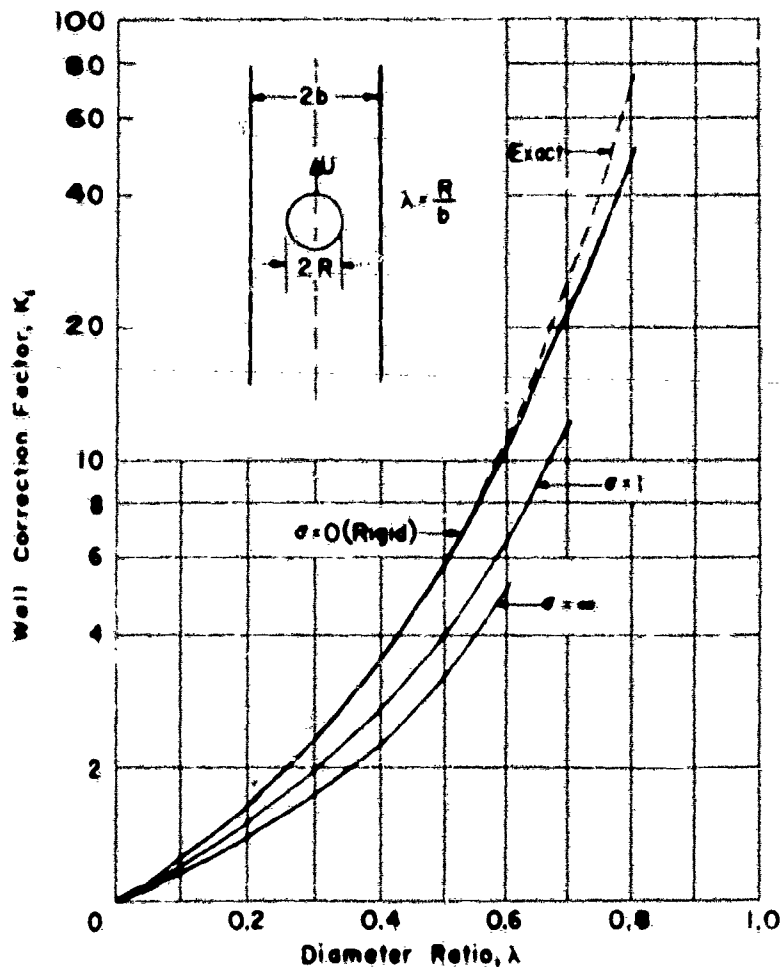


Figure 13 – Wall Correction Factors K_1 for Spheres within a Cylinder

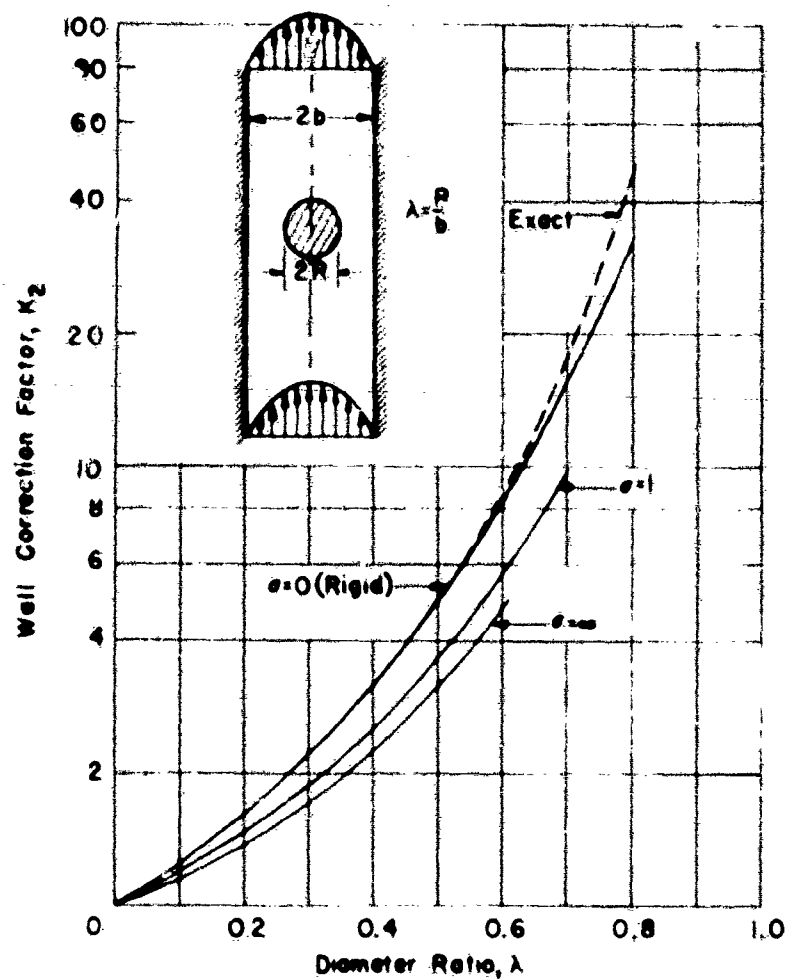


Figure 14 – Wall Correction Factors K_2 for Spheres within a Cylinder

$$K_1 = \frac{1 - 0.75857 \frac{1-\sigma}{1+2/3\sigma} \lambda^5}{1 - 2.1050 \frac{1+2/3\sigma}{1+\sigma} \lambda + 2.0865 \frac{1}{1+\sigma} \lambda^3 - 1.7068 \frac{1-2/3\sigma}{1+\sigma} \lambda^5 + 0.72603 \frac{1-\sigma}{1+\sigma} \lambda^6} \quad [75]$$

$$K_2 = \frac{1 - \frac{2}{3} \frac{1}{1+2/3\sigma} \lambda^2 - 0.20217 \frac{1-\sigma}{1+2/3\sigma} \lambda^5}{1 - 2.1050 \frac{1+2/3\sigma}{1+\sigma} \lambda + 2.0865 \frac{1}{1+\sigma} \lambda^3 - 1.7068 \frac{1-2/3\sigma}{1+\sigma} \lambda^5 + 0.72603 \frac{1-\sigma}{1+\sigma} \lambda^6} \quad [94]$$

thus

$$\text{Drag} = 6\pi\mu R \frac{1+2/3\sigma}{1+\sigma} (UK_1 - VK_2) \quad [95]$$

The wall correction factors K_1 and K_2 (as obtained from Equations [75] and [94]) are given as a function of the diameter ratio in Figures 13 and 14 for three viscosity ratios ($\sigma = 0$ [rigid case], 1, and ∞). The above solutions for the wall correction factors for the fluid spheres are estimated to be valid for diameter ratios up to 0.5.

The expression for the exterior stream function for fluid spheres in Poiseuille flow is given by Equation [87] with coefficients b_0 and a_1 determined from Equation [89]. The stream function in the interior region is given by Equation [73] (with coefficients as given in Equation [87] and determined from Equation [89]).

TERMINAL VELOCITIES OF SPHERES

In many applications it is of interest to determine the terminal velocity (U) of rising or falling spheres. Such a velocity can easily be determined from the results of the present investigation. A body rising or falling under the influence of gravity reaches such a velocity (terminal velocity) when all forces acting on it are in equilibrium:

$$\text{Drag} + \text{Buoyant Force} + \text{Weight} = 0$$

or

$$6\pi\mu R \frac{1 + \frac{2}{3}\sigma}{1 + \sigma} (UK_1 - VK_2) - \frac{4}{3}\pi R^3 \gamma g + \frac{4}{3}\pi R^3 \gamma_i g = 0 \quad [96]$$

where γ is the density of liquid and γ_i is the density of sphere; hence

$$U = \frac{2}{9} \frac{R^2 g (\gamma - \gamma_i)}{\mu K_1} \frac{1 + \sigma}{1 + \frac{2}{3}\sigma} + V \frac{K_2}{K_1} \quad [97]$$

for $\sigma = 0$, $V = 0$, and $K_1 = 1$; i.e., rigid spheres moving in a still infinite medium, Equation [97] reduces to Stokes law; namely,

$$U = \frac{2}{9} \frac{R^2 g (\gamma - \gamma_i)}{\mu}$$

The value of the appropriate wall correction factor is obtained from Figure 5 or Table 1, Figure 10 and Figure 14 or Table 2. The ratio K_2/K_1 appearing in Equation [97] has been evaluated from Equations [75] and [94]; thus

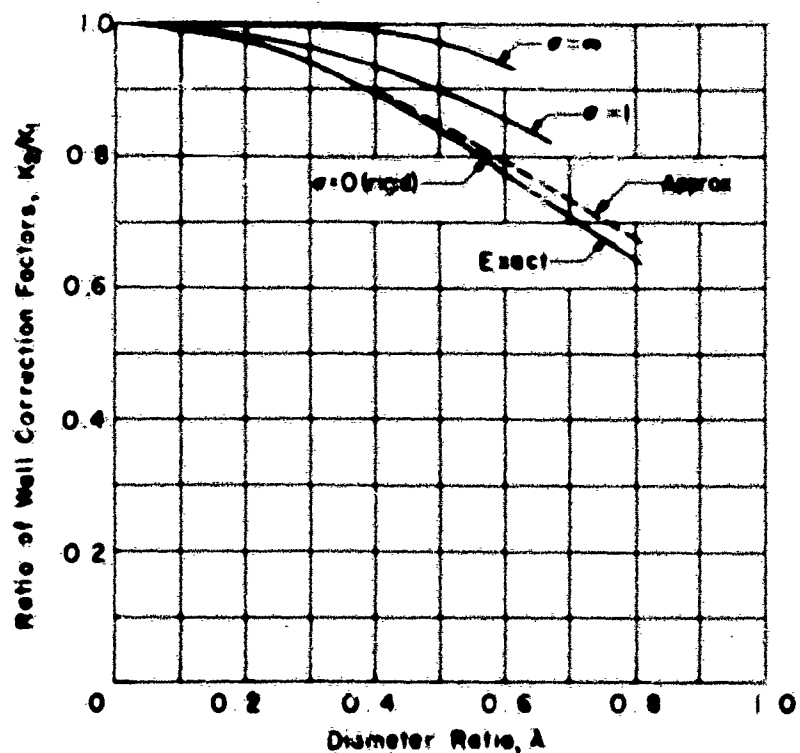


Figure 15 - Wall Correction Factor Ratio

$$\frac{K_2}{K_1} = 1 - \frac{\frac{2}{3} \frac{1}{1 + \frac{2}{3}\sigma} \lambda^5 - 0.55640 \frac{1 - \sigma}{1 + \frac{2}{3}\sigma}}{1 - 0.75857 \frac{1 - \sigma}{1 + \frac{2}{3}\sigma} \lambda^5} \quad [98]$$

The ratio K_2/K_1 is given in Figure 15 for a range of diameter ratios λ for viscosity ratios $\sigma = 0$, 1, and ∞ . For the rigid case ($\sigma = 0$), the ratio K_2/K_1 obtained from the exact solution is also included.

SUMMARY

The steady, slow motion of rigid and fluid spheres translating along the axis of an infinitely long cylinder has been examined. The solutions are given in terms of the stream function. An exact solution for the motion of rigid spheres is obtained in terms of an infinite set of linear algebraic equations for the coefficients in the stream function. The drag of a sphere in motion within a moving liquid is shown to be composed of two parts: namely, the drag due to the motion of the sphere in a still liquid inside the cylindrical tube, and the drag due to the motion of the liquid inside the cylindrical tube past a stationary sphere:

$$\text{Drag} = 6\pi\mu R(UK_1 - VK_2)$$

The drag experienced by rigid spheres has been determined for the two special cases over a range of ratios of sphere-to-cylinder diameter. A very good approximation for the drag of the rigid spheres is obtained by utilizing the first two equations of the infinite set. For fluid spheres, an approximate solution (similar to the rigid case) has been obtained.

The approximate expressions for the wall correction factors for the spheres in a cylindrical tube take the form

$$K_1 = \frac{1 - 0.75857 \frac{1-\sigma}{1+2/3\sigma} \lambda^5}{1 - 2.1050 \frac{1+2/3\sigma}{1+\sigma} \lambda + 2.0865 \frac{1}{1+\sigma} \lambda^3 - 1.7068 \frac{1-2/3\sigma}{1+\sigma} \lambda^5 + 0.72803 \frac{1-\sigma}{1+\sigma} \lambda^6}$$

$$K_2 = \frac{1 - \frac{2}{3} \frac{1}{1+2/3\sigma} \lambda^2 - 0.20217 \frac{1-\sigma}{1+2/3\sigma} \lambda^5}{1 - 2.1050 \frac{1+2/3\sigma}{1+\sigma} \lambda + 2.0865 \frac{1}{1+\sigma} \lambda^3 - 1.7068 \frac{1-2/3\sigma}{1+\sigma} \lambda^5 + 0.72803 \frac{1-\sigma}{1+\sigma} \lambda^6}$$

Experimental results for the rigid and fluid cases confirm the theory. In general, the results show that the wall effect for the fluid spheres is less than for corresponding rigid spheres.

FUTURE INVESTIGATIONS

Since the experimental investigation showed that fluid bodies deform into spheroids (ellipsoids of revolution), work is currently in progress to obtain a solution for the stream function in spheroidal coordinates. It is hoped that an exact solution for fluid bodies might be obtained in this fashion, or it might at least improve the present approximate solution for large diameter ratios.

ACKNOWLEDGMENTS

The authors wish to express their thanks to Mr. R.D. Cooper, formerly with the David Taylor Model Basin, for checking many of the basic equations, and to Professor C.A. Shreeve, Jr., of the University of Maryland, who supervised the thesis work.

APPENDIX A

GEGENBAUER POLYNOMIALS $C_n^{-1/2}(t)$

The Gegenbauer polynomials $C_n^{-1/2}(t)$ which appear in the solution for the stream function in spherical coordinates can be evaluated by making use of their relationship to the Legendre polynomials (Reference 25, page 77):

$$C_n^{-1/2}(t) = \frac{1}{2n-1} [P_{n-2}(t) - P_n(t)]$$

where $P_n(t)$ is the Legendre polynomial of degree n , and

$$t = \cos \theta.$$

$$C_0^{-1/2} = 1^*$$

$$C_1^{-1/2} = -t$$

$$C_2^{-1/2} = \frac{1}{2} (1 - t^2)$$

$$C_3^{-1/2} = \frac{1}{2} (1 - t^2) t$$

$$C_4^{-1/2} = \frac{1}{8} (1 - t^2) (5t^2 - 1)$$

$$C_5^{-1/2} = \frac{1}{8} (1 - t^2) (7t^2 - 3) t$$

$$C_6^{-1/2} = \frac{1}{16} (1 - t^2) (21t^4 - 14t^2 + 1)$$

$$C_7^{-1/2} = \frac{1}{16} (1 - t^2) (33t^4 - 30t^2 + 5) t$$

$$C_8^{-1/2} = \frac{1}{128} (1 - t^2) (429t^6 - 495t^4 + 135t^2 - 5)$$

$$C_9^{-1/2} = \frac{1}{128} (1 - t^2) (715t^6 - 1001t^4 + 385t^2 - 35) t$$

$$C_{10}^{-1/2} = \frac{1}{256} (1 - t^2) (2431t^8 - 4004t^6 + 2002t^4 - 298t^2 + 7)$$

etc.

*For $n = 0, 1$, the polynomials were evaluated from the expansion given on page 78, Reference 25.

$$(1-t^2) \frac{d^2 C_n^{-1/2}(t)}{dt^2} = -\frac{n(n-1)}{2n-1} (P_{n-2} - P_n) = -n(n-1) C_n^{-1/2}$$

$$\frac{d C_n^{-1/2}(t)}{dt} = -P_{n-1}$$

APPENDIX B

EVALUATION OF INTEGRALS CONTAINING $K_0(a\rho)$ AND $K_1(a\rho)$

The integrals containing $K_0(a\rho)$ were evaluated by successive differentiation with respect to x , starting with the known integral (Reference 25, page 35)

$$\int_0^\infty K_0(a\rho) \cos ax \, da = \frac{\pi}{2} \frac{1}{r}$$

Hence

$$\int_0^\infty \rho^2 K_0(a\rho) \cos ax \, da = \frac{\pi}{2} r \sin^2 \theta = \pi r C_2^{-1/2}(t)$$

$$\int_0^\infty \rho^2 K_0 a \sin ax \, da = \frac{\pi}{2} \sin^2 \theta \cos \theta = \pi C_3^{-1/2}(t)$$

$$\int_0^\infty \rho^2 K_0 a^2 \cos ax \, da = \frac{\pi}{2} \frac{1}{r} \sin^2 \theta (-3 \cos^2 \theta + 1) = \frac{2}{5} \pi \frac{1}{r} C_2^{-1/2}(t) - \frac{12}{5} \pi \frac{1}{r} C_4^{-1/2}(t)$$

$$\int_0^\infty \rho^2 K_0 a^3 \sin ax \, da = \frac{\pi}{2} \frac{1}{r^2} \sin^2 \theta (-15 \cos^2 \theta + 9) \cos \theta$$

$$\int_0^\infty \rho^2 K_0 a^4 \cos ax \, da = \frac{\pi}{2} \frac{1}{r^3} \sin^2 \theta (105 \cos^4 \theta - 90 \cos^2 \theta + 9)$$

$$\int_0^\infty \rho^2 K_0 a^5 \sin ax \, da = \frac{\pi}{2} \frac{1}{r^4} \sin^2 \theta (945 \cos^4 \theta - 1050 \cos^2 \theta + 225) \cos \theta$$

$$\int_0^\infty \rho^2 K_0 a^6 \cos ax \, da = \frac{\pi}{2} \frac{1}{r^5} \sin^2 \theta (-10,395 \cos^6 \theta + 14,175 \cos^4 \theta - 4725 \cos^2 \theta + 225)$$

$$\int_0^\infty \rho^2 K_0 a^7 \sin ax \, da = \frac{\pi}{2} \frac{1}{r^6} \sin^2 \theta (-135,135 \cos^6 \theta + 218,295 \cos^4 \theta - 99,225 \cos^2 \theta + 11,025) \cos \theta$$

$$\int_0^\infty \rho^2 K_0 a^8 \cos ax \, da = \frac{\pi}{2} \frac{1}{r^7} \sin^2 \theta (2,027,025 \cos^8 \theta - 3,783,780 \cos^6 \theta + 2,182,950 \cos^4 \theta - 896,900 \cos^2 \theta + 11,025)$$

etc.

The integrals containing $K_1(a\rho)$ were evaluated by successive differentiation with respect to x , starting with the known integral²⁶

$$\int_0^\infty K_1(a\rho) \sin ax da = \frac{\pi}{2} \frac{x}{\rho} \frac{1}{r}$$

Hence

$$\int_0^\infty \rho K_1 a \cos ax da = \frac{\pi}{2} \frac{1}{r} \sin^2 \theta = \frac{\pi}{r} C_2^{-1/2}(t)$$

$$\int_0^\infty \rho K_1 a^2 \sin ax da = \frac{\pi}{2} \frac{1}{r^2} \sin^2 \theta \cdot 3 \cdot \cos \theta = \frac{3\pi}{r^2} C_3^{-1/2}(t)$$

$$\int_0^\infty \rho K_1 a^3 \cos ax da = \frac{\pi}{2} \frac{1}{r^3} \sin^2 \theta (-15 \cos^2 \theta + 3) = -\frac{12\pi}{r^3} C_4^{-1/2}(t)$$

$$\int_0^\infty \rho K_1 a^4 \sin ax da = \frac{\pi}{2} \frac{1}{r^4} \sin^2 \theta (-105 \cos^2 \theta + 45) \cos \theta = -\frac{60\pi}{r^4} C_5^{-1/2}(t)$$

$$\int_0^\infty \rho K_1 a^5 \cos ax da = \frac{\pi}{2} \frac{1}{r^5} \sin^2 \theta (945 \cos^4 \theta - 630 \cos^2 \theta + 45) = \frac{360\pi}{r^5} C_6^{-1/2}(t)$$

$$\int_0^\infty \rho K_1 a^6 \sin ax da = \frac{\pi}{2} \frac{1}{r^6} \sin^2 \theta (10,395 \cos^4 \theta - 9450 \cos^2 \theta + 945) \cos \theta$$

$$\int_0^\infty \rho K_1 a^7 \cos ax da = \frac{\pi}{2} \frac{1}{r^7} \sin^2 \theta (-135,135 \cos^6 \theta + 62,370 \cos^4 \theta - 36,855 \cos^2 \theta + 945)$$

$$\int_0^\infty \rho K_1 a^8 \sin ax da = \frac{\pi}{2} \frac{1}{r^8} \sin^2 \theta (-2,027,025 \cos^6 \theta + 1,621,620 \cos^4 \theta - 654,885 \cos^2 \theta + 84,105) \cos \theta$$

etc.

Another group of integrals is derivable from

$$\int_0^\infty K_0(a\rho) \sin ax da = \frac{1}{r} \ln \frac{x+r}{\rho} = \frac{1}{2r} \ln \frac{1+\cos \theta}{1-\cos \theta}$$

and

$$\int_0^\infty K_1(a\rho) a \sin ax da = -\frac{\rho}{r^3} \ln \frac{x+r}{\rho} - \frac{1}{r\rho} + \frac{\rho}{r^2(x+r)} = -\frac{1}{r^2} \left(\frac{1}{2} \sin \theta \ln \frac{1+\cos \theta}{1-\cos \theta} + \frac{\cos \theta}{\sin \theta} \right)$$

These integrals, however, result in infinite velocities for $\rho = 0$; hence the integrals

$\int_0^\infty K_0(a\rho) a^n \cos az da$ (where $n = \text{odd}$), $\int_0^\infty K_0(a\rho) a^n \sin az da$ (where $n = \text{even}$),

$\int_0^\infty K_1(a\rho) a^n \cos az da$ (where $n = \text{even}$), and $\int_0^\infty K_1(a\rho) a^n \sin az da$ (where $n = \text{odd}$) cannot

be used in the present problem. These integrals correspond to $P_n(t)$ -terms in the spherical coordinate solution.

APPENDIX C

NUMERICAL EVALUATION OF INTEGRALS S_2^n AND S_4^n

The integrals S_2^n and S_4^n defined as

$$S_2^n = \int_0^\infty \frac{(ab)^{n-1}}{I_1^2(ab) - I_0(ab) I_2(ab)} d(ab)$$

$$S_4^n = \int_0^\infty (ab)^n \frac{I_1(ab) K_1(ab) + I_2(ab) K_0(ab)}{I_1^2(ab) - I_0(ab) I_2(ab)} d(ab)$$

were evaluated using Simpson's rule. Increments of 0.1 in the argument were used for the range 0 to 3; for larger values increments of 0.2 were used. The values of the Bessel functions were obtained from Reference 27.

The $S_3^1 + 2S_4^0$ had to be evaluated in the combined form; namely,

$$S_3^1 + 2S_4^0 = - \int_0^\infty \frac{2(I_1 K_1 + I_2 K_0) - 1}{I_1^2 - I_0 I_2} d(ab)$$

The same increments as above were used. The range 0 to 0.1 was evaluated using the small value approximation of

$$- \int_0^{0.1} [1 - 2\gamma + 2 \ln 2 - 2 \ln(ab)] d(ab)$$

where γ = Euler's constant (0.577215665 ...).

As check on the computations the integral $S_3^3 + 2S_4^0$ was also evaluated in the combined form.

The evaluated integrals are tabulated in Table 3.

TABLE 3

Values of the Integrals S_2^n , S_4^n , and $S_3^n + 2S_4^{n-1}$

n	S_2^{n+1}	S_4^n	$S_3^{n+1} + 2S_4^n$
0	-	-	- 4.40866
2	17.87328	6.55507	- 4.76313
4	60.5861	15.3046	- 29.9769
6	675.6858	133.8245	- 408.0368
8	137.7658×10^2	226.4366×10	$- 924.7838 \times 10$
10	431.9631×10^3	609.0951×10^2	$- 310.1441 \times 10^3$
12	191.2120×10^5	235.5148×10^4	$- 144.1090 \times 10^5$
14	112.1989×10^7	123.3180×10^6	$- 875.3534 \times 10^6$

APPENDIX D

EVALUATION OF THE DRAG OF SPHERES FROM THE STREAM FUNCTION

The drag of the spheres is to be evaluated by integrating the forces on the surface of the sphere in the x -direction

$$D = D_\theta + D_r \quad [D-1]$$

where D is the drag,

D_θ is the tangential drag, and

D_r is the normal drag.

From Figure 16 these components are:

$$D_\theta = \int p_{r,\theta} \sin \theta \, dS \quad [D-2]$$

$$D_r = - \int p_{rr} \cos \theta \, dS$$

where dS = surface element = $2\pi R^2 \sin \theta \, d\theta$.

Hence

$$D_\theta = 2R^2 \pi \int_0^\pi p_{r,\theta} \sin^2 \theta \, d\theta \quad [D-3]$$

$$D_r = -2R^2 \pi \int_0^\pi p_{rr} \sin \theta \cos \theta \, d\theta$$

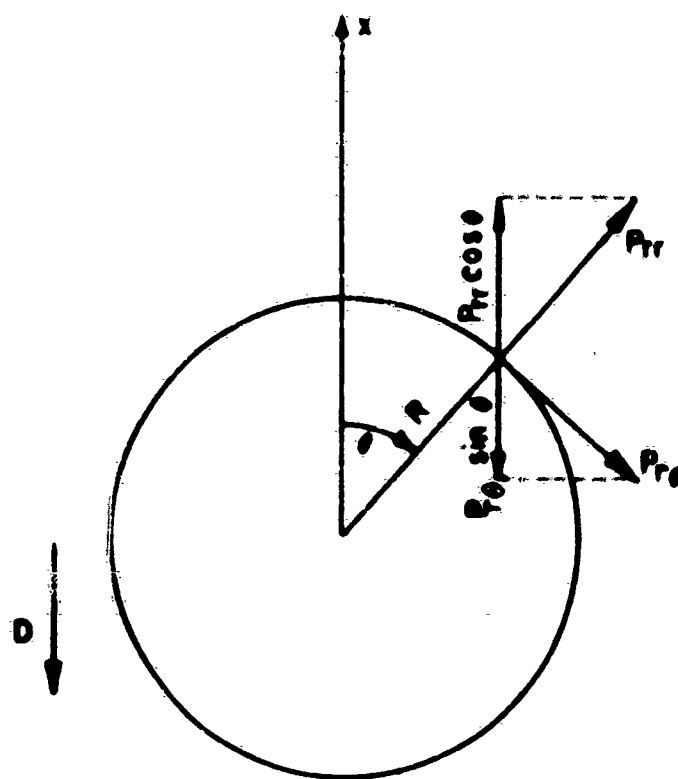


Figure 16 -- Stress Components on Surface of Sphere

From Reference 18, page 374:

$$p_{r,\theta} = \mu \left(\frac{1}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} \right) \quad \text{Tangential stress} \quad [D-4]$$

$$p_{rr} = -p + 2\mu \frac{\partial v_r}{\partial r} \quad \text{Normal stress} \quad [D-5]$$

Utilizing the expressions for the velocity components (Equations [18]), the tangential stress becomes:

$$\begin{aligned} \frac{1}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} = \sum_2^\infty \frac{C_n^{-1/2}}{\sin \theta} \left[2n(n-2) A_n r^{n-3} + 2(n^2-1) B_n \frac{1}{r^{n+2}} + 2(n^2-1) C_n r^{n-1} + \right. \\ \left. + 2n(n-2) D_n \frac{1}{r^n} \right] \end{aligned} \quad [D-6]$$

Since $C_n^{-1/2} = (1/2n-1)(P_{n-2} - P_n)$, the tangential drag becomes

$$D_\theta = 2R^2 \pi \mu \int_0^\pi \sum_2^\infty \frac{2}{2n-1} (P_{n-2} - P_n) \left[n(n-2) A_n R^{n-3} + (n^2-1) B_n \frac{1}{R^{n+2}} + (n^2-1) C_n R^{n-1} + n(n-2) D_n \frac{1}{R^n} \right] \sin \theta d\theta \quad [D-7]$$

From Reference 25, page 52

$$\int_0^\pi P_n(\cos \theta) \sin \theta d\theta = 2 \quad \text{for } n = 0$$

$$= 0 \quad \text{otherwise;}$$

hence

$$D_\theta = 8R^2 \pi \mu \left(B_2 \frac{1}{R^4} + C_2 R \right) \quad [D-8]$$

In order to evaluate the normal stress component (p_{rr}), the pressure (p) must first be obtained from the equation of motion. From Reference 18, page 373, we have for slow flow:

$$\frac{1}{\mu} \frac{\partial p}{\partial r} = \nabla^2 v_r - 2 \frac{v_r}{r^2} - 2 \frac{\cot \theta}{r^2} v_\theta - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} \quad [D-9]$$

$$\frac{1}{\mu} \frac{\partial p}{r \partial \theta} = \nabla^2 v_\theta - \frac{v_\theta}{r^2 \sin^2 \theta} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \quad [D-10]$$

where $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{\cot \theta}{r^2} \frac{\partial}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$.

$$\frac{\partial v_r}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + 2 \frac{v_r}{r} + \frac{\cot \theta}{r} v_\theta = 0 \quad \text{Equation of Continuity} \quad [D-11]$$

Hence

$$\frac{2}{r} \left(\frac{\partial v_r}{\partial r} + \frac{v_r}{r} \right) = - \left(\frac{1}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r^2} + \frac{\cot \theta}{r^2} v_\theta \right) \quad [D-12]$$

and

$$\frac{1}{\mu} \frac{\partial p}{\partial r} = \nabla^2 v_r + \frac{2}{r} \left(\frac{\partial v_r}{\partial r} + \frac{v_r}{r} \right) \quad [D-13]$$

Hence from Equation [18]

$$\frac{1}{\mu} \frac{\partial p}{\partial r} = \sum_2^{\infty} P_{n-1} \left[-2(2n+1) C_n r^{n-2} + 2(2n-3) D_n \frac{1}{r^{n+1}} \right] \quad [D-14]$$

Integrating

$$p = \mu \sum_2^{\infty} P_{n-1} \left[-\frac{2(2n+1)}{n-1} C_n r^{n-1} - \frac{2(2n-3)}{n} D_n \frac{1}{r^n} \right] + f(\theta) \quad [D-15]$$

Integrating the equation containing the partial derivative of p with respect to θ , we obtain

$$p = \mu \sum_2^{\infty} P_{n-1} \left[-\frac{2(2n+1)}{n-1} C_n r^{n-1} - \frac{2(2n-3)}{n} D_n \frac{1}{r^n} \right] + f(r) \quad [D-16]$$

hence $f(r) = f(\theta) = 0 \quad n \geq 2$.

Therefore

$$p_{rr} = \mu \sum_2^{\infty} P_{n-1} \left\{ -2(n-2) A_n r^{n-3} + 2(n+1) B_n \frac{1}{r^{n+2}} + \left[\frac{2(2n+1)}{n-1} - 2n \right] C_n r^{n-1} + \left[\frac{2(2n-3)}{n} + 2(n-1) \right] D_n \frac{1}{r^n} \right\} \quad [D-17]$$

From Reference 25, page 52,

$$\int_0^\pi P_m \sin 2\theta d\theta = 0 \quad \text{for } m \geq 2 \text{ or } m \text{ even}$$

$$= \frac{4}{3} \quad \text{for } m < 2 \text{ and } m \text{ odd.}$$

The normal drag becomes:

$$v_r = -4R^2 \pi \mu \left(2B_2 \frac{1}{R^4} + 2C_2 R + D_2 \frac{1}{R^2} \right) \quad [D-18]$$

Adding the two drag components we obtain for the drag of a sphere moving in a medium:

$$D = -4\pi \mu D_2 \quad [D-19]$$

The same result has been obtained by Savic²⁰ for the case $A_n = C_n = 0$.

APPENDIX E

MOTION OF SPHERES IN A SPHERICAL CONTAINER

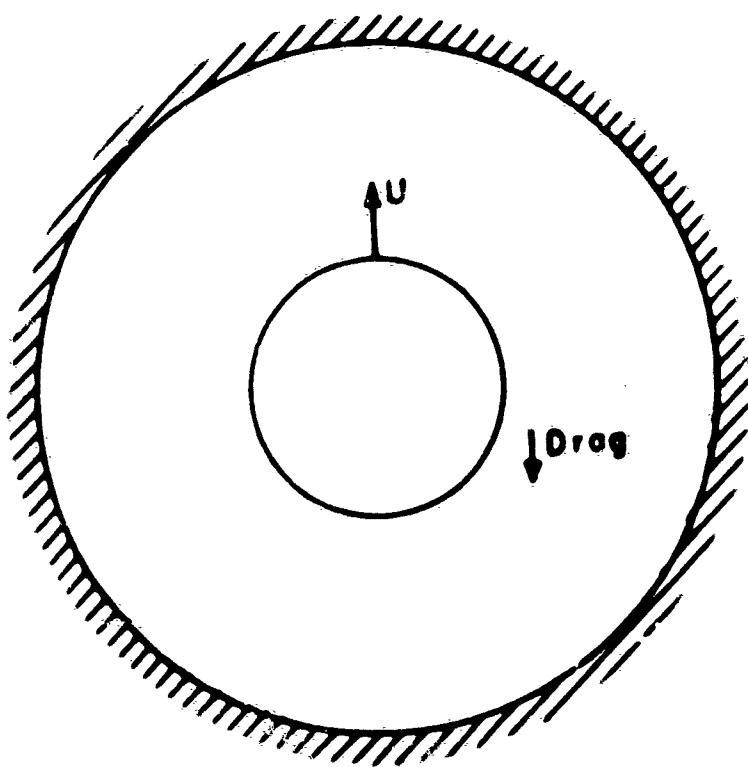
The motion of a sphere at the instant it passes the center of a spherical container (neglecting unsteady effects)* is of interest as guide and upper bound for the motion within a cylindrical container, since the wall effects for corresponding spherical boundaries exceed those for cylindrical ones.**

The origin of the coordinates is taken at the center of the sphere, and the container is assumed to be moving at constant velocity U in the negative x -direction, as indicated in Figure 17. The boundary conditions for the sphere have been given in a previous section. The boundary conditions at the container walls ($r = P$) are:

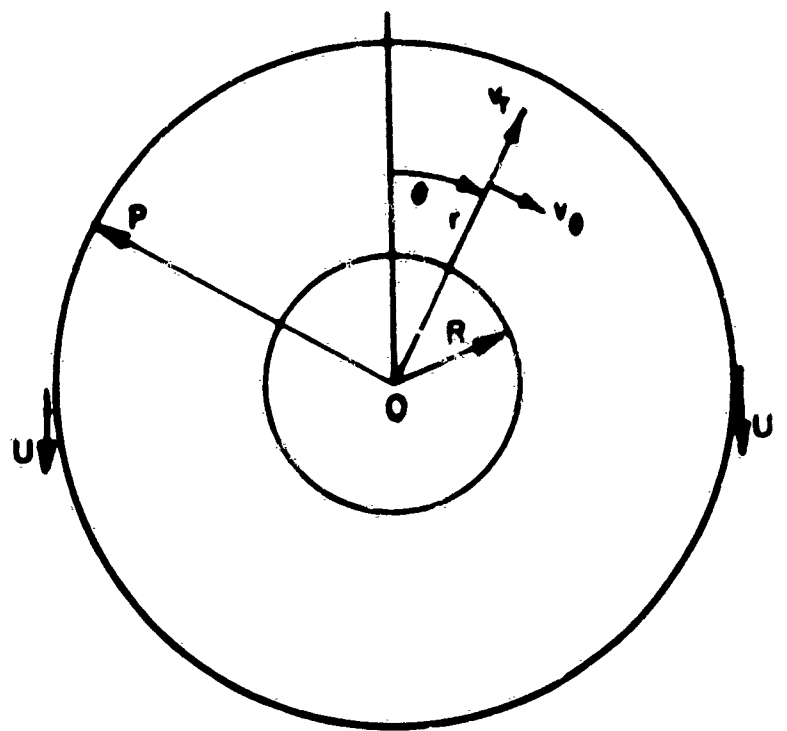
$$\begin{aligned} v_r &= -U \cos \theta \quad \text{or} \quad \Psi = \frac{1}{2} U P^2 \sin^2 \theta \\ v_\theta &= U \sin \theta \end{aligned} \quad [E-1]$$

From Equations [E-1], [39], and [67] we obtain:

$$\begin{aligned} A_n P^n + B_n \frac{1}{P^{n-1}} + C_n P^{n+2} + D_n \frac{1}{P^{n-3}} - [UP^2]_{n=2} &= 0 \\ n A_n P^{n-2} - (n-1) B_n \frac{1}{P^{n+1}} + (n+2) C_n P^n - (n-3) D_n \frac{1}{P^{n-1}} - [2U]_{n=2} &= 0 \end{aligned} \quad [E-2]$$



Sphere Passing Center of Spherical Container
with Velocity U



Equivalent System: Sphere Fixed; Spherical
Container Moving with Velocity U

Figure 17 - Definition Sketch for Sphere in Spherical Container

*The corresponding two-dimensional problem for a rigid cylinder has been stated by Frazer (Reference 28).

**See Reference 29.

Solving for A_n, B_n, C_n, D_n, E_n , and F_n , we obtain

$$\begin{aligned}
 A_2 &= \frac{U}{R^2} \frac{1 + \sigma + \frac{5}{4} \lambda^3 - \frac{3}{2} \left(\frac{3}{2} - \sigma \right) \lambda^5}{1 + \sigma - \frac{3}{2} \left(\frac{3}{2} + \sigma \right) \lambda + \frac{5}{2} \lambda^3 - \frac{3}{2} \left(\frac{3}{2} - \sigma \right) \lambda^5 + (1 - \sigma) \lambda^6} \\
 B_2 &= \frac{1}{2} UR^3 \frac{1 - (1 - \sigma) \lambda^3}{1 + \sigma - \frac{3}{2} \left(\frac{3}{2} + \sigma \right) \lambda + \frac{5}{2} \lambda^3 - \frac{3}{2} \left(\frac{3}{2} - \sigma \right) \lambda^5 + (1 - \sigma) \lambda^6} \\
 C_2 &= -\frac{1}{2} \frac{U}{R^2} \frac{\left(\frac{8}{2} + \sigma \right) \lambda^3 - \frac{3}{2} \lambda^5}{1 + \sigma - \frac{3}{2} \left(\frac{3}{2} + \sigma \right) \lambda + \frac{5}{2} \lambda^3 - \frac{3}{2} \left(\frac{3}{2} - \sigma \right) \lambda^5 + (1 - \sigma) \lambda^6} \\
 D_2 &= -\frac{3}{2} UR \frac{1 + \frac{2}{3} \sigma - (1 - \sigma) \lambda^5}{1 + \sigma - \frac{3}{2} \left(\frac{3}{2} + \sigma \right) \lambda + \frac{5}{2} \lambda^3 - \frac{3}{2} \left(\frac{3}{2} - \sigma \right) \lambda^5 + (1 - \sigma) \lambda^6} \\
 E_2 &= -\frac{U}{2} \frac{\left(1 - \frac{5}{2} \lambda^3 + \frac{3}{2} \lambda^5 \right) \sigma}{1 + \sigma - \frac{3}{2} \left(\frac{3}{2} + \sigma \right) \lambda + \frac{5}{2} \lambda^3 - \frac{3}{2} \left(\frac{3}{2} - \sigma \right) \lambda^5 + (1 - \sigma) \lambda^6} \\
 F_2 &= \frac{U}{2} \frac{1}{R^2} \frac{\left(1 - \frac{5}{2} \lambda^3 + \frac{3}{2} \lambda^5 \right) \sigma}{1 + \sigma - \frac{3}{2} \left(\frac{3}{2} + \sigma \right) \lambda + \frac{5}{2} \lambda^3 - \frac{3}{2} \left(\frac{3}{2} - \sigma \right) \lambda^5 + (1 - \sigma) \lambda^6} \\
 A_n &= B_n = C_n = D_n = E_n = F_n = 0 \quad n \geq 4
 \end{aligned} \tag{E-3}$$

The stream function in the exterior of the sphere thus takes the form

$$\Psi = \frac{1}{2} \sin^2 \theta \left(A_2 r^2 + C_2 r^4 + B_2 \frac{1}{r} + D_2 r \right) \tag{E-4}$$

where A_2, B_2, C_2 , and D_2 are given above, whereas the stream function in the interior becomes:

$$\Psi_i = \frac{1}{2} \sin^2 \theta \left[E_2 r^2 \left(1 - \frac{r^2}{R^2} \right) \right] \tag{E-5}$$

where E_2 is given in Equation [E-3].

For an infinite medium $D_2 = -\frac{3}{2} UR \frac{1 + 2/3 \sigma}{1 + \sigma}$. Hence the wall correction factor

for motion in a spherical container becomes

$$K = \frac{1 - \frac{1-\sigma}{1+2/3\sigma} \lambda^5}{1 - \frac{9}{4} \frac{1+2/3\sigma}{1+\sigma} \lambda + \frac{5}{9} \frac{1}{1+\sigma} \lambda^3 - \frac{9}{4} \frac{1-2/3\sigma}{1+\sigma} \lambda^5 + \frac{1-\sigma}{1+\sigma} \lambda^6} \quad [\text{E-6}]$$

For a rigid sphere ($\sigma = 0$) the above equation reduces to:

$$K = \frac{1 - \lambda^5}{1 - \frac{9}{4} \lambda + \frac{5}{9} \lambda^3 - \frac{9}{4} \lambda^5 + \lambda^6} \quad [\text{E-7}]$$

For a fluid sphere of vanishing viscosity ($\mu_i = 0$ or $\sigma = \infty$):

$$K = \frac{1 + \frac{3}{2} \lambda^5}{1 - \frac{3}{2} \lambda + \frac{3}{2} \lambda^5 - \lambda^6} \quad [\text{E-8}]$$

For a fluid sphere with viscosity equal to that of the internal medium ($\sigma = 1$):

$$K = \frac{1}{1 - \frac{15}{8} \lambda + \frac{3}{4} \lambda^3 - \frac{3}{8} \lambda^5} \quad [\text{E-9}]$$

The wall correction factors for these three cases are shown as a function of λ in Table 4 and Figure 18.

TABLE 4 – Wall Correction Factors for Spheres Moving in a Spherical Container

λ	Rigid Spheres	Fluid Spheres	
		$\mu_i = \mu$	$\mu_i = 0$
0.0	1.000	1.000	1.000
0.1	1.286	1.229	1.176
0.2	1.756	1.575	1.428
0.3	2.573	2.126	1.815
0.4	4.106	3.066	2.469
0.5	7.294	4.831	3.722
0.6	14.948	8.636	6.569
0.7	37.830	18.762	14.826
0.8	138.224	58.480	50.773
0.9	1209.778	431.779	439.156

*This equation agrees with that of Cunningham⁷ and Lee⁹ except that Cunningham shows a $-\lambda^6$ term; this appears to be a printing error.

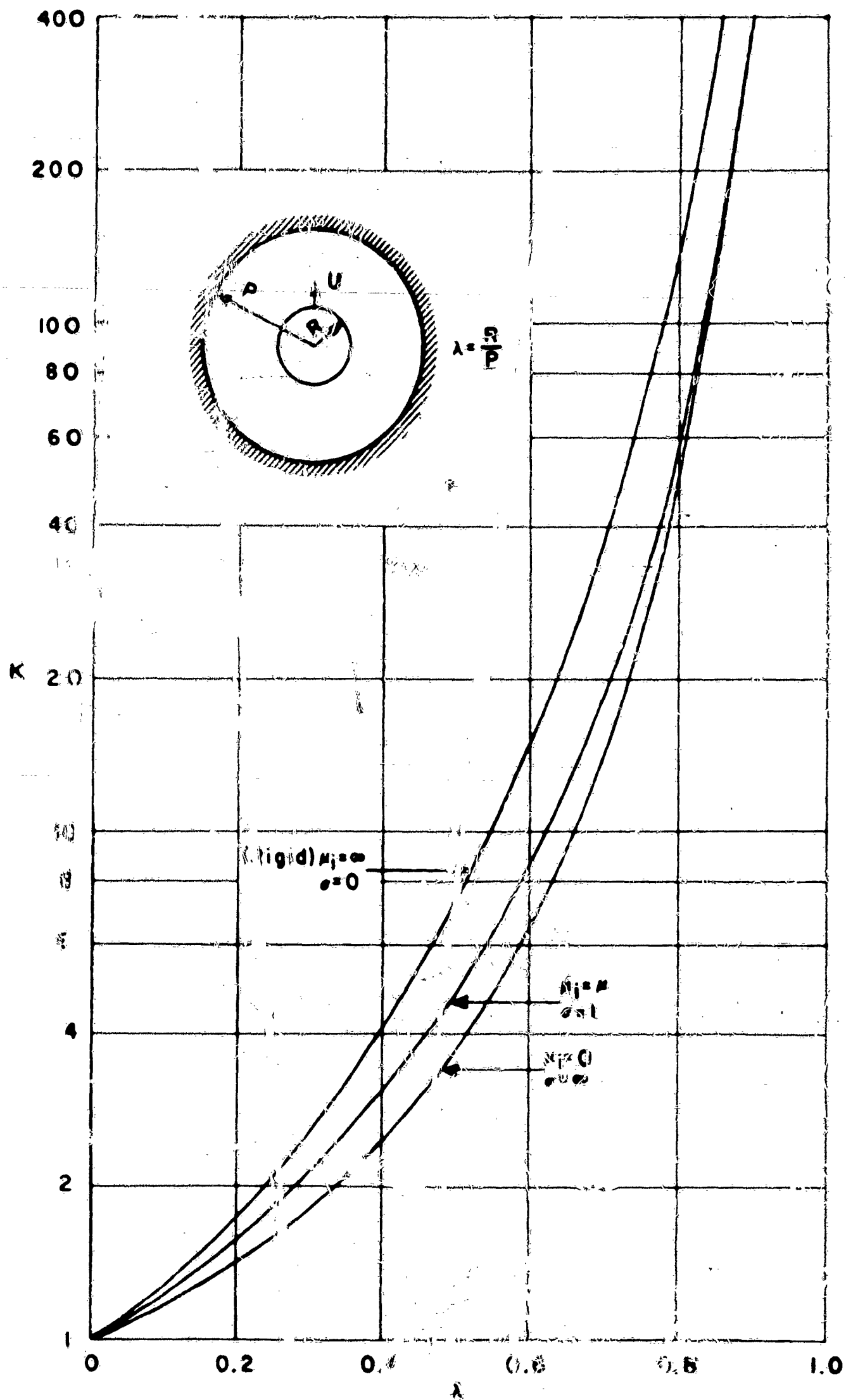


Figure 18 — Wall Correction Factors for Spheres Moving in a Spherical Container

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